N-PERSON DIFFERENTIAL GAMES AND MULTICRITERION OPTIMAL CONTROL PROBLEMS

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Dedicated to

my Parents and my Sisters

CERTIFICATE

Certified that this work, "N-Person Differential Games and Multicriterion Optimal Control Problems" by U.R. Prasad, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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LIST OF SYMBOLS AND NOTATION

We employ vector matrix notation generally. Vectors and matrices are denoted by single letters. Transpose Notation is not used generally where it is clear from the context. However, it is used in the linear-quadratic problems since the results are familiar in this way. A vector is Positive or Negative or Zero if all its components are so respectively. Similarly a vector is Less Than (is Below in a vector space representation) another if each of its components is less than (below) the corresponding components of the other. This applies to other inequalities as well. Variable subscripts indicate partial derivatives. The Notation used in the Thesis is indicated in the text. However we append here most of the symbols used.

(1) VARIABLES AND PARAMETERS

t,τ	the independent variable (time)
0	a scalar parameter
σ	a scalar or vector parameter
x,ξ	state vector
x_σ	system unknown parameter vector
y	observation vector
u	control vector
w	white noise vector

λ adjoint variable vector

Ψ adjoint variable matrix (used in the Matrix Minimum Principle)

ξ reference vector in the tracking problem

(2) CONSTANTS

a number of segments in the Terminal Surface

b, c system constants in the Double Integral Plant formulation

α, η constants in the switching surface representation

β constant in the Value Function representation

δ small scalar quantity

ϵ small scalar or vector quantity

k, λ_0 constants associated with the adjoint variables

l dimensionality of the control variable constraints

m dimensionality of the observation vector

n dimensionality of the state vector

r dimensionality of the control action vector

ρ dimensionality of the white noise vector

μ constant vector appearing in the Minimum Principle because of the control variable constraints

ϑ constant vector appearing in the Transversality Condition because of the terminal constraints

M penalty associated with the terminal constraint satisfaction

N number of players in a general game

(3) ELEMENTS OF VECTOR SPACES

e vector in the Euclidean space

r, s, x vector in the state space

z vector in the cost space

q vector in the cost-state space

(4) MATRICES

$\left.\begin{matrix}A\\B\\C\\D\end{matrix}\right\}$	formulation of the linear-quadratic problems dynamic and observation equations
F	matrix weighting the terminal error
Q	matrix weighting the error on the trajectory
R	matrix weighting the control action
P	error covariance matrix
S, Z	matrices satisfying Riccati equations appearing in the solution
Θ	covariance of the white noise vector

(5) FUNCTIONS AND FUNCTIONALS

f	appearing in the state equation
g	defined in the solution of the tracking problem
h	observation function
H	Hamiltonian
I	Supercriterion
J	performance index or criterion functional
K	constraint function on the control and state variables
L	loss function
T, X	parametric representation of the terminal surface
U	strategy
ψ	terminal constraint function
V	cooperative Value Function
W	noncooperative Value Function
ϕ	terminal cost function
Φ^{ℓ}	derivatives of H_u with supercript denoting the order

(6) SETS, REGIONS, SURFACES AND CURVES

Ω	control restraint set
G	regions in the noncooperative solution of the Double Integral Plant
R	regions in the cooperative solution of the Double Integral Plant
$\mathcal{C}$	cost orthant defined in the Minimum Principle for the Pareto optimal solution
Σ	class of admissible strategies
$\mathcal{U}$	class of playable strategies in the Normal Form
$\mathcal{M}, \mathcal{N}, \mathcal{J}, \mathcal{R}$	occurs in the Regular Decomposition of the Playing Space
E	set of Equilibrium Points
M	set of Minimax Points
Γ, γ, Λ	switching curves

(7) SUPERSCRIPTS AND SUBSCRIPTS

_	underbar used when similar quantities of all the players are put together
o	superscript used to refer Pareto optimal or Nash cooperative solutions
o	subscript refers to the initial time
f	subscript to indicate final time
*	superscript to refer equilibrium solutions
p	superscript to denote a general player
T	superscript to denote transpose
i,j	subscripts refer to the Regular Decomposition
t	superscript to denote cumulated observations
$[t_o, t_f]$	subscript to denote the interval of definition

(8) SPECIAL NOTATION

N	neighbourhood
E	expectation with suffix denoting the space over which it is taken
$\prod_i$	multiplication symbol with the multiplication index
$\sum_j$	summation symbol and summation index
$\dot{(\ )}$	denotes differentiation
$\Vert\ \Vert_Q$	norm with respect to the matrix Q
$C^{(k)}$	class of functions which have continuous derivatives upto order k
$\in$	belongs to

SYNOPSIS

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N-PERSON DIFFERENTIAL GAMES AND
MULTICRITERION OPTIMAL CONTROL PROBLEMS

Deterministic two-person zero-sum Differential Games were first studied by Isaacs, mainly in connection with Pursuit-Evasion problems. Berkovitz gave a rigorous mathematical foundation to this subject. Recently N-Person Differential Games have started to receive attention. In N-Person Games, the payoff function vector of the players orders only partially their joint strategies and hence the solution of these games require the invoking of Supercriteria. In this sense, N-Person Games are related to problems of Vector Programming and Decision-Making under Uncertainty. Extension of these interrelationships to a multistage setting under dynamic constraints to study N-Person Differential Games and Multicriterion Optimal Control Problems provides the motivation for this Thesis.

N-Person Differential Games admit various Information Patterns and levels of Cooperation among

the players. The solution concepts of Finite Games are generally applicable to these games. Thus the Noncooperative Solution is given in terms of Equilibrium and Minimax points, and the Cooperative Solution is given by Pareto Optimal Points. The application of these concepts in the case of deterministic Differential Games is discussed in Chapter II.

In Chapter III, Necessary Conditions, similar to the minimum principle of Pontryagin, are derived for the Noncooperative Solution in terms of Equilibrium Points. It is established that the optimal strategies of the players necessarily induce Equilibrium Points in the Hamiltonians, one for each player, with the usually associated Euler-Lagrange equations and the Transversality and Corner conditions.

A general N-Person Differential Game exhibits a variety of switching surfaces similar to those encountered in two-person zero-sum games. These can be broadly classified into Transition, Singular, Dispersal and Abnormal Surfaces. While the Transition Surfaces are obtained by the application of corner conditions, the others require further conditions for their determination. For example, the Singular Surfaces are constructed by the application of the Legendre-Clebsch condition in its generalized form. Chapter IV is devoted to a discussion of these surfaces and their construction.

The concept of Pareto Optimality is discussed in detail in Chapter V along with the Cooperative Solutions of Differential Games. The necessary conditions for Pareto Optimality show that Pareto Optimal Solutions are obtained by solving a class of parametrized optimal control problems with the criterion functionals given as convex combinations of the payoff functionals of all the players. The various cooperative solutions differ in the underlying Supercriteria to single out one Pareto optimal point as the solution. It is shown that Multicriterion Optimal Control Problems can be solved as Cooperative N-Person Differential Games without sidepayments and with equal information to all the players. A computational method is suggested for the resulting Nash Cooperative Solution. The solution of the double integral plant with the twin objectives of minimizing time and fuel is given as a running example of a two-person nonzero-sum game throughout the Thesis.

The study of Imperfect and Incomplete Information Differential Games is initiated by considering Multicriterion Optimal Control under Uncertainty in Chapter VI. Because of their equal information feature, these problems are easier to solve than the corresponding Differential Games. A thorough study of such problems requires additional mathematical concepts from Stochastic Optimal Control and Markov Processes and is suggested for further investigation.

CHAPTER I

INTRODUCTION

1.1 GENERAL

It has long been recognized that the Theory of Games, which initially arose as a mathematical reduction of Competitive Economic Behaviour (von Neumann and Morgenstern 1944), has potential application in many-sided complex decision-making problems of great significance. While a Mathematical Programming Problem can be viewed as a simple decision-making problem, Statistical Decision Theory deals with the more complex problem of optimization under uncertainty. Two-person zero-sum game theory is applied for the solution of this problem, with the Uncertainty playing the role of an antagonist to the Decision-maker (Blackwell and Girschik 1954).

The Theory of Control Processes can be considered as dealing with multistage decision-making problems under dynamic constraints and the Dynamic Programming of Bellman (1957) is essentially developed for such problems. Pontryagin (1962) gave necessary conditions to the deterministic optimal control problem which in turn can be considered a generalization of the calculus of variations (Hestenes 1966). The interrelationships between these

topics made Control Theory one of the major disciplines of Applied Mathematics in recent times.

As uncertainty enters into the formulation of a control problem in terms of ignorance of the plant dynamics and uncontrollable inputs to the plant (Horowitz 1963), the problem of control under uncertainty received the attention of many investigators both under deterministic and stochastic formulations. In either case, the problem is cast as a two-person zero-sum game only recently (Ragade and Sarma 1967 and Sworder 1966).

In any practical control system, it is inherent that various basically different requirements are to be met in the design. The present Optimal Control Theory is restrictive in the assumption that these objectives can be reduced to a single criterion for minimization. Zadeh (1963) suggested the use of a vector criterion, with its components representing the various requirements, to judge the performance of a system. The resulting Multicriterion Optimal Control Problem (see Chang 1966) is a multistage generalization of the Vector Programming Problem (Kuhn and Tucker 1951 and Da Cunha and Polak 1967) under dynamic constraints.

The seemingly different problems, sketched above, have basic similarities. In this chapter, after giving

a brief review of N-person games, we introduce the Multicriterion Optimal Control Problems. We then study the interrelationships between the problems of games, decision-making under uncertainty and vector programming.

The objective of this Thesis is to pursue these aspects further to a multistage setting and study N-Person Differential Games and Multicriterion Optimal Control Problems under basically similar frameworks.

1.2 BRIEF REVIEW OF N-PERSON GAMES

The foundations of the mathematical theory of Games of Strategy were laid by von Neumann. Along with Morgenstern (1944), he emphasized a new approach to Competitive Economic Behaviour through a mathematical reduction to suitable Games of Strategy with N participants in general.

N-person games permit different information patterns to the players and various levels of cooperation among them (Luce and Raiffa 1957). However the study of these games as initiated by von Neumann and pursued by others later on is largely in the Normal Form which does not permit explicit information patterns to the players and the game looses its multistage character in this form.

The main contribution in Game Theory is the min-max theorem for the two-person zero-sum game. N-person

games with cooperation and sidepayments permitted between the players are studied in terms of the Characteristic Function Theory (von Neumann and Morgenstern 1944 and Luce and Raiffa 1957). While the solution of the two-person zero-sum game in terms of saddle points is appealing, the Characteristic Function Theory is unsatisfactory in many respects (Luce and Raiffa 1957). The Solutions may be numerous and 'embarassingly rich' on the one hand, while on the other a recent result by Lucas (1967) shows that the Solution may not even exist for some classes of games. Alternative approaches to the concept of Solution are the Value of Shapley, Reasonable Outcomes of Milner and -Stability (Luce and Raiffa 1957).

A further approach to N-person games is due to Nash (1951), who defines the solution of a game with no cooperation between the players in terms of its Equilibrium Points. By arguing that the noncooperative model is more basic since any cooperation between the players can be represented as moves in a larger game, he obtains the cooperative solution of a two-person game without side-payments as the noncooperative solution of the overall game (Nash 1953). The ideas and results of Nash are fruitfully extended by Harsanyi (1956, 1959 and 1964) to cover larger classes of games, connecting them up with the

earlier theories of Bargaining in Econometrics. A different approach to the N-person cooperative game without sidepayments is due to Aumann and Peleg (1960) based on a Generalized Characteristic Function.

Further extensions of the above results have been numerous as can be seen in (Kuhn and Tucker 1950 and 1953, Dresher et.al. 1957, Tucker and Luce 1959 and Dresher et.al. 1964) and the references compiled in them. Infinite games, in which the players are permitted infinite number of strategies, of which the games of timing and the games on the unit square are examples (see Karlin 1959), are studied by many authors.

Kuhn (1953) and Dalkey (1953) introduced the study of games in Extensive Form which can explicitly take care of the imperfectness of the information to the players. A player's information is perfect if and only if he possesses at every move the information regarding the history of his previous moves and those of the others.

A particular class of the infinite games with a continuum of strategies and a continuum of moves are the Differential Games, the two-person zero-sum form of which have been studied extensively by Isaacs (1965), Berkovitz (1964 and 1967) and others (see Ho et.al. 1969 for an excellent compilation of References in the area of

Differential Games). Since the formulation of Differential Games follows on the same lines as the optimal control problems except for two or more controlling agencies or players, they can be considered as a generalization to the optimal control problems and they retain their multistage character. Pontryagin (1966) and others (Patsyukov 1968) formulated a class of Differential Games with a heirarchical form of information to the players. Recently there has been interest in Differential Games with N-players (Starr and Ho 1969, Prasad and Sarma 1969 and Case 1969).

While imperfect information games are studied in the literature (Kuhn 1953 and Dalkey 1953) with extensions to Differential Games (Behn and Ho 1968, Rhodes and Luenberger 1969, Rhodes 1969 and Sarma et.al. 1969), incomplete information games have not received much attention except the recent contributions by Harsanyi (1967 and 1968) and Ragade (1968). A player has complete information if and only if he has complete knowledge of the strategic capabilities and objectives of his and other players in the game.

Under a very general framework of a positional game, Ragade (1968) establishes the relation between the various game models discussed earlier and the multistage games such as Stochastic Games, Recursive Games etc. It

is also interesting to note that there have been attempts to consider a game with infinite number of players (Kalisch and Nering 1959 and Kannai 1964).

1.3 INTRODUCTION TO MULTICRITERION OPTIMAL CONTROL

It is well known that the state-space method of controller design has the advantage of a more realistic mathematical formulation of the overall control problem involving the essential dynamics and constraints and the tasks, the system is supposed to perform. Thus the system state x satisfies a vector differential equation

$$\dot{x} = f(x, u, t) \tag{1.1}$$

where the control input u is chosen from the restraint set Ω. The system state is transferred from the initial condition $x(t_o) = x_o$ to a terminal surface specified by

$$\psi(x_f, t_f) = 0 \tag{1.2}$$

To choose from the variety of control inputs which achieve the task satisfying the specified constraints, a performance index or a criterion functional, depending upon the requirements or the objectives to be met by the system, is minimized with respect to the control policies. The criterion functional is given by

$$J^p[x_o, t_o, u] = \phi^p(x_f, t_f) + \int_{t_o}^{t_f} L^p(x,u,t)\, dt \tag{1.3}$$

If J^p is a scalar, it introduces a total ordering on the

control policies and the optimal policy which minimizes J^p is chosen. Of course there can be associated problems of existence and uniqueness of solutions (Lee and Markus 1967).

However there are usually requirements, basically different in nature, that enter into the judgement of the system performance and the choice of a scalar functional is necessarily arbitrary and subjective. One way to overcome this difficulty is to propose multiple criteria i.e., with $p = 1, \ldots N$ in (1.3). The vector $\underline{J} = (J^1, \ldots J^N)$ is called the criterion functional vector. Now a system may be better than another with respect to some components of $\underline{J}$ and worse with respect to others. In other words, the vector criterion cannot induce a natural (or total) ordering on the set of allowable control policies. We describe some of the earlier methods of designing a system given multiple criteria.

Nelson (1964) specifies some acceptable bounds, similar to the classical design techniques, on all but one criterion. The resulting problem is an optimal control problem with respect to the free criterion, with several isoperimetric constraints. A second method due to Chyung (1967) orders all the criteria according to their importance and the optimization is performed in a heirarchy. In other words, each criterion is applied on

the optimal controls resulting from the application of the preceding criterion. Lastly, the vector criterion is reduced to a single index by attaching suitable positive weights to the various components of the vector. Many problems may not permit the assumptions in these methods.

Now, supposing there is a control law such that the performance of the system cannot be improved with respect to any component of the vector criterion without simultaneously deteriorating the performance with respect to the other components, then such a control law obviously has a Weak Optimality Characteristic. Such control laws are called Noninferior and in general are nonunique. Without imposing Supercriteria, there does not seem to be a way of choosing a particular noninferior policy as the solution of the problem. The earlier methods discussed in the preceding paragraph can be considered as the application of such Supercriteria.

If uncertainty arises in a control problem in whatever form (Horowitz 1963), it can be characterized, to a large extent, in a statistical sense. If such a characterization is complete and the performance index is scalar valued, one can find a control policy which is best in the Statistical Expectation Sense. In the case when the characterization of the uncertainty is incomplete, the

performance index cannot induce a total ordering on the set of control policies, even if the performance index is scalar-valued. Once again, a solution to this problem is obtained only through the application of a Supercriterion (Sworder 1966). In the next section, we study the interrelationships between the problems of Vector Programming, Games and Decision-making under Uncertainty and the application of Supercriteria to solve them.

1.4 VECTOR PROGRAMMING, GAMES AND UNCERTAINTY

The problem of vector minimization arose in many branches of Science and Mathematics. The concept of Noninferiority discussed in the earlier section is related to the concepts of Admissibility in Statistics (Wald 1950), Efficient Solutions in Economics (Karlin 1959), Pareto Optimality in Game Theory (Luce and Raiffa 1957) and Minimal Solution in Vector Mathematical Programming (Kuhn and Tucker 1951). All these signify Weakly Optimal Solutions arising as minimal solutions with respect to a partial ordering introduced by the vector function on the set of allowable control or decision variables. Since in general such solutions are nonunique, this may be referred to as the Dilemma introduced by the partial ordering. To resolve this Dilemma, Supercriteria have to be applied to single out the Solution to the Problem from the variety of Noninferior Solutions.

In the case of a cooperative game, the solution which is noninferior should reflect the strategic potentialities of the players. For this, the noncooperative solution of the game is studied extensively such as the powers of the players in forming coalitions, issuing threats to the other players etc. (Luce and Raiffa 1957). Thus the Supercriterion used in obtaining the cooperative solution depends upon the noncooperative solution. This procedure is made possible because each player's decision variables in a game are known.

In a general vector minimization problem, such a procedure of obtaining the Supercriterion through the noncooperative solution is not possible straight away. However, if the various decision variables are allocated to the different components of the vector function, the method applies. This is equivalent to writing, for example in (1.1) and (1.3), as

$$u = (u^1, \ldots u^N) \tag{1.4}$$

The solution obtained by such a procedure would have the advantage of reflecting the tradeoff factors between the various criteria in a game-theoretic sense for the considered allocation. In many problems, the allocation can be done in a natural manner and this can be exploited fruitfully.

In a two-person zero-sum game, no cooperation between the two players is possible. In a Decision-making Problem under Uncertainty, the dilemma introduced by the uncertainty can be resolved by considering uncertainty as an intelligent antagonist. Such a solution would be optimal under the worst case and is the basis of all worst-case designs. Once again the procedure can be thought of as the application of a Supercriterion and is borrowed from Game Theory.

The problem of vector minimization under uncertainty can similarly be reduced to a two-person zero-sum game with a vector payoff function. The concepts of Approachability and Excludability by Blackwell (1956) are Weakly Optimal and the solution should be contained in sets having these properties.

1.5 OUTLINE OF THE THESIS

We give, in the next chapter, a general framework to study N-Person Differential Games and discuss the applicability of the concepts of Finite Games to the Differential Games.

In Chapter III, Necessary and some Sufficient Conditions are presented for the Noncooperative Solution of the game along with simple illustrative examples. This is followed, in Chapter IV, with the general construction of various switching surfaces encountered in these games.

An example of a Double Integral Plant is worked out in detail to obtain the noncooperative solution. Many phenomena, both encountered in Finite Games and otherwise, are shown.

In Chapters V and VI, we extend the inter-relationships studied in Section 1.4 to the multistage case and consider Multicriterion Optimal Control Problems with and without Uncertainty. It is shown, that these problems can be solved as N-Person Cooperative Differential Games without Sidepayments and with equal information to all the players. In Chapter V, we study the deterministic problems and in Chapter VI, problems under uncertainty along with some of the current problems of interest in Optimal Control Theory. Chapter VII concludes the Thesis.

CHAPTER II

BASIC STRUCTURE OF N-PERSON DIFFERENTIAL GAMES

2.1 INTRODUCTION

In this chapter we develop a general framework to study N-person differential games. First we review the solution concepts of N-person games, some of which are already mentioned in Chapter I. Most of these concepts are associated with the Normal Form of the game and depend mainly whether the game is played under noncooperation or cooperation among the players. Though the Noncooperative Solution can be considered in its own right, it is also necessary in some form for the Cooperative Solution. We follow the references (Nash 1951 and Harsanyi 1964) for the Noncooperative Solution and (von Neumann and Morgenstern 1944, Nash 1953, Harsanyi 1959 and Luce and Raiffa 1957) for the Cooperative Solution.

A fairly general class of Deterministic N-Person Differential Games is formulated and extensions to other classes are indicated in Section 2.3. This is followed up with some basic solution concepts in these games. In essence, we show the adaptability of the solution concepts of finite games to the present situation and discuss the implications bearing in mind the dynamics of the problem.

2.2 CONCEPTS FROM THE THEORY OF N-PERSON GAMES

An N-person game arises out of a situation involving a Conflict of Interest among a set of N players and Game Theory deals with the Decision-Making Problems of each of the players in this situation. The structure of the conflict, consisting of the Rules of the Game, is best represented in a model known as the game in its Extensive Form.

Games in Extensive Form:

A game in its extensive form consists of a Topological Tree (see Figure 2.1), with a distinguished vertex (the hollow node O in the figure) representing the first move and a Payoff Function defined for each player on the terminal vertices (the hollow nodes f_1, f_2 ... f_{14}))representing the Outcomes of the game. Further, the nonterminal vertices are partitioned into (N+1) sets corresponding to the N players and Nature[1] called the Player Sets (labelled with 1, 2, 3, 4 for the players and O for the Nature[2]). Each player set is further subdivided into Information Sets for the respective player (shown by broken lines in the figure), the nodes in which are

1 In a Completely Deterministic Game, there will be no player set corresponding to Nature.

2 The first move may belong to any player in general. In Figure 2.1 it is Nature's move.

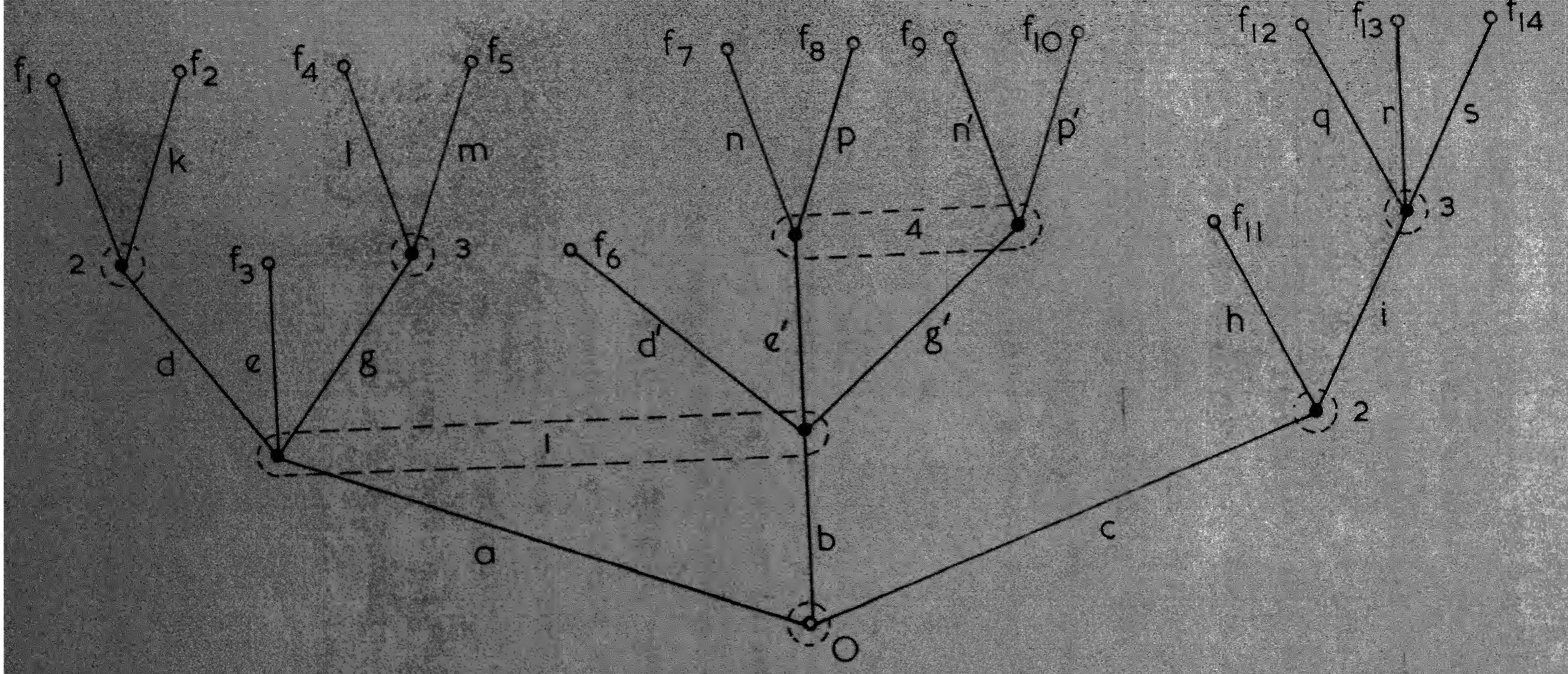

FIG. 2.1 AN EXAMPLE OF A GAME IN ITS EXTENSIVE FORM

Indistinguishable in terms of the Previous Transitions and Current Choices available to him. The nodes represent Moves in general, while the branches at a node corresponding to a player (say a, b and c at node O of Nature) represents the Alternatives available to him and the Transitions resulting from each Choice of the Alternatives (Indistinguishable Choices are represented by the same letter in the figure, for example d d', e e' and g g' for player 1). A probability distribution governs the transitions at a Nature's Move.

Information Patterns to Players:

A player is said to have Perfect Information if all of his Information Sets contain only one node (Players 2 and 3 have Perfect Information in Figure 2.1). Otherwise his information is Imperfect (for example Players 1 and 4 in the figure). While imperfect information is effectively portrayed by the extensive form, there is another concept known as Complete Information which is not represented in the Extensive Model. Under this assumption, a player has Complete Information if he has full knowledge of the game in its Extensive Form and a Complete Information Game implies Complete Information to all its players.

The objective of each player is to minimize[3] his

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3 To confirm with control theoretic terminology.

payoff function and this assumption goes under the name of Rationality in game theory. The other assumptions, made outside the Extensive Model, are the constraints regarding the Communication and Cooperation between the players.

Normal Form of a Game:

A Choice Function mapping any player's information sets into his corresponding alternatives at the moves in the information set is called a Pure Strategy of the player. Given the game in its extensive form, one can enumerate all the pure strategies for the various players. A game in its Normal Form is given by the pure strategies and the payoff functions of the players as shown below:

$$\mathcal{G} \triangleq \{\mathcal{U}^1, \ldots \mathcal{U}^N;; J^1, \ldots J^N\} \tag{2.1}$$

where $\mathcal{U}^1, \ldots \mathcal{U}^N$ are the players' strategy sets and $J^1, \ldots J^N$ are the payoff functions. Since a selection of one strategy each by the players determines an outcome, $J^1, \ldots J^N$ can be considered as scalar real-valued functions on the product of strategy sets. That is for $p = 1, \ldots N$,

$$J^p : \mathcal{U}^1 \times \mathcal{U}^2 \ldots \mathcal{U}^N \rightarrow \mathbb{R} \tag{2.2}$$

The game obtained by replacing each $\mathcal{U}^p$ by the set of all probability distributions on $\mathcal{U}^p$ is called the Mixed Extension of $\mathcal{G}$ and each of the probability distributions is called a Mixed Strategy. Against this, a different type of random strategy, called a Behaviour Strategy, is defined

by specifying a probability distribution at every information set into the corresponding alternatives.

In its normal form, the game looses its multi-move character and the imperfectness of information to the various players. Full knowledge of all the sets $\mathcal{U}^1, \ldots \mathcal{U}^N$ and the functions $J^1, \ldots J^N$ by a player is regarded as complete information to him in this model. Most of the solution concepts are studied for Complete Information Games in Normal Form.

The Solution to a Game consists in finding Optimal Strategies for each player to minimize his payoff function taking into account his and other players' information patterns and payoff functions and other constraints on communication and cooperation between the players. The various solution concepts are basically different and incomplete. This defect in game theory seems to be due to the lack of a game model which includes the constraints on communication and cooperation between the players.

Noncooperative Solution of a Game[4]:

The noncooperative solution of a game in normal form is given in terms of its Nash Equilibrium and Minimax

4 We assume complete information in what follows unless otherwise specified.

Strategies (Nash 1951, Harsanyi 1959 and Luce and Raiffa 1957). The game in (2.1) is said to have an Equilibrium Point if strategies $(U^{1*}, \dots U^{N*})$ exist such that the following hold for $p = 1, \dots N$.

$$J^p(U^{1*}, \dots U^{p-1*}, U^p, U^{p+1*}, \dots U^{N*}) \geq J^p(U^{1*}, \dots U^{N*}) \qquad (2.3)$$

or equivalently

$$\min_{U^p \in \mathcal{U}^p} J^p(\underline{U}^* ; U^p) = J^p(\underline{U}^*) \qquad (2.4)$$

where

$$(\underline{U}^* ; U^p) = (U^{1*}, \dots U^{p-1*}, U^p, U^{p+1*}, \dots U^{N*}) \qquad (2.5)^5$$

and $$(\underline{U}^*) = (U^{1*}, \dots U^{N*}) \qquad (2.6)^5$$

Thus no player can unilaterally deviate from his equilibrium strategy and improve his payoff function. Nash (1950) proved the existence of such points in mixed strategies for finite games. The equilibrium points of a Two-Person Zero-Sum Game are called its Saddle Points.

In a two-person zero-sum game a player can insure himself a Security Level for his payoff by playing a certain pure strategy - known as his Minimax Strategy - and his antagonist cannot prevent him from doing this even if he has full knowledge of the above strategy. Once again this

5 This notation is used freely hereafter.

latter assumption which is made in a worst-case sense is outside the structure of the game and if mixed strategies are allowed, saddle point strategies are always to be preferred by the players to their minimax strategies. In a N-person game, minimax strategies have the same significance for any player with the rest of the players acting as a combined antagonist.

Since it is possible to have more than one equilibrium point in a general game, extra concepts are needed to define the noncooperative solution. Two equilibrium points $\underline{U}^*$ and $\hat{\underline{U}}$ are Equivalent if for all $p = 1, \ldots N$, we have

$$J^p(\underline{U}^*) = J^p(\hat{\underline{U}}) \tag{2.7}$$

Two equilibrium points $\underline{U}^*$ and $\hat{\underline{U}}$ are Interchangeable if any Recombination $\tilde{\underline{U}}$ - every $\tilde{U}^p$ is either U^{p*} or $\hat{U}^p$ - is also an equilibrium point. All the saddle points are automatically both equivalent and interchangeable in the case of two-person zero-sum games and hence constitute the solution. This being not true in a general N-person Game, the solution is defined differently depending upon whether the players are allowed to communicate between themselves to decide on certain equilibrium strategies or not. The Vocal Solution (V-Solution), in which communication is allowed, is given either as a set E of equilibrium

points Safely Equivalent[6] with respect to some admissible set A containing E or as a set M of minimax points. The Tacit Solution (T-Solution), in which no communication is allowed, is given as a set M^* of minimax points or as a set E^* of Interchangeable equilibrium points.

The interchangeability is essential only for the T-solution. Otherwise there will be a coordination problem since there is no communication between the players in this case. The safe equivalence is a weakened equivalence concept which is as follows. The Safe Payoff for player p in the set A of equilibrium points, from strategy U^p is given by

$$J^{p*}(U^p\ ;\ A) = \min_{\underline{U} \in A(U^p)} J^p(\underline{U}) \tag{2.8}$$

where $A(U^p)$ is the set of those equilibrium points in A where player p uses U^p as his strategy. Two equilibrium points $\underline{U}^*$ and $\hat{\underline{U}}$ are Safely Equivalent with respect to A if their respective safe payoffs are equal for all the players. That is for $p = 1, \ldots N$, we have

$$J^p(\underline{U}^*\ ;\ A) = J^p(\hat{\underline{U}}\ ;\ A) \tag{2.9}$$

The set A in the V- and T-solutions itself is obtained by certain reduction procedures applied on the set of all equilibrium points in the game based on some Payoff and

---

6 To be defined below.

Risk Dominance relations. Payoff Dominance shows the preferences of the players over the several equilibrium strategies and Risk Dominance compares the risks, each of the players takes in sticking to a certain equilibrium strategy favourable to him against the possibility of the other players not adhering to it. These concepts are fairly involved to be presented in detail here. The V- and T-solutions are given as minimax points if one fails to determine the solution in terms of equilibrium points by the above procedure.

Cooperative Solutions:

The cooperative solution of a game is given by its Pareto Optimal Strategies. Under uninhibited communication[7] between them, the players under cooperation agree to play a binding Pareto optimal strategy and enforce it mutually. A joint strategy $\underline{U}^o$ is Pareto Optimal if for any other strategy $\underline{U}$, we have

$$\left\{\begin{matrix} J^p(\underline{U}) \leqslant J^p(\underline{U}^o) \\ p = 1,\ldots N \end{matrix}\right\} \text{ only if } \left\{\begin{matrix} J^p(\underline{U}) = J^p(\underline{U}^o) \\ p = 1,\ldots N \end{matrix}\right\} \qquad (2.10)$$

The strategy being jointly coordinated, it implies that the information sets corresponding to this strategy are obtained by the total information available to all the players put together. However, while implementing, each player will take recourse to his own information sets.

7 In the Vocal Solution under noncooperation, the players do not strike compromise outside the equilibrium points or minimax strategies.

Expressed mathematically, the vector $\underline{J} = (J^1, \ldots J^N)$
n only introduce a Partial Ordering on the space of joint
rategies and minimality with respect to this ordering is
reto optimality. Thus while Nash equilibrium is a Weak
tability Concept (since the strategy is stable for any player
gainst his unilateral deviation), Pareto optimality is a
eak Optimality Concept in N-person game theory.

Since in general there are several Pareto Optimal
strategies in a game, the cooperative solution involves a
selection of one such strategy. If the payoffs to the
various players can be compared and the total payoff
redistributed among them, then that Pareto optimal strategy
which minimizes their Total Payoff expressed in a common unit
is chosen. The Characteristic Function Theory (von Neumann
and Morgenstern 1953 and Luce and Raiffa 1957) deals with the
Redistribution Problem by considering the Security Levels
of the various Coalitions in the game.

In games in which a comparison of payoffs cannot be
made, the Dilemma is resolved by considering Bargaining
between the players with the initial point as the noncooper-
ative solution. This solution is justified by Nash (1953)
by an axiomatic approach as well as considering the Threats
and Demands for the players as moves in an overall game. The
noncooperative solution reflects the Optimal Threats while
the bargaining problem reflects the Optimal Demands.

2.3 FORMULATION OF N-PERSON DIFFERENTIAL GAMES

Unlike finite games, Differential Games cannot be represented in the form of a game tree since each player possesses a continuum of moves and a continuum of alternatives at each move. Thus the advantage of representing imperfect information pictorially by information sets is lost. The node becomes the state of the game in this case with the first node specified by an Initial Condition and the outcomes described by a suitable Terminal Surface. The alternatives for any player at each move are specified by a Control Restraint Set and the transition occuring because of a choice of the alternatives is described by a Differential Equation. Any imperfectness of information to a player is represented by an Observation Equation. Thus the following is the Extensive Formulation of a Deterministic N-Person Differential Game.

The state of the game, x of dimension n, satisfies a vector differential equation

$$\dot{x} = f(x, \underline{u}, t) \qquad (2.11)$$

where

$$\underline{u} = (u^1, \ldots u^p, \ldots u^N) \qquad (2.12)$$

and u^p is the r^p - dimensional control action vector of the p^{th} player and t denotes time.

The state of the game is to be transferred by the
ntrol actions of the players from an initial state

$$x(t_o) = x_o \tag{2.13}$$

a final state contained in a terminal surface of
mension n given by

$$\psi(x_f, t_f) = 0 \tag{2.14}$$

parametrically as

$$x_f = X(\sigma) \quad ; \quad t_f = T(\sigma) \tag{2.15}$$

iere σ ranges over an n-dimensional cube.

The p^{th} player chooses his control action
$^p \in \Omega^p$, over the time interval $[t_o, t_f]$ so as to
inimize his payoff functional

$$^p[x_o, t_o, \underline{u}] = \phi^p(x_f, t_f) + \int_{t_o}^{t_f} L^p(x, \underline{u}, t)\, dt \tag{2.16}$$

he control action of each player is based on his informa-
ion of the state of the game specified through his
bservation equation.

The p^{th} player makes the m^p - dimensional(with
$^p < n$) observations y^p given by

$$y^p = h^p(x, t) \tag{2.17}$$

n this context, perfect information to a player implies
that his observations are identical with the state at any

time. The p^{th} player chooses his strategy as a function of his information into his control restraint set Ω^p, i.e.,

$$u^p = U^p(y^p, t) \tag{2.18}$$

Other assumptions about the smoothness properties of the various functions, the region of the state space where the game takes place and sofforth will be introduced later on.

Many new classes of games can be constructed by considering that the functions such as f, h^p and L^p are noisy. That is we have

$$\dot{x} = f(x, \underline{u}, w_1, t) \tag{2.19}$$

$$y^p = h^p(x, w_2^p, t) \tag{2.20}$$

where w_1 and w_2^p are random disturbance vectors. In this case, the players minimize their payoff functions in a statistical expectation sense. These are called Stochastic Differential Games with imperfect information because of the presence of noise terms in (2.19) and (2.20). These and other information patterns are considered in recent literature (Ragade 1968, Ciletti 1969 and Rhodes 1969).

A player is said to have complete information if he has full knowledge about the various functions involved in the formulation as well as the statistical information about

all the random disturbances. The solution concepts in these games will be discussed next.

2.4 SOLUTION CONCEPTS OF N-PERSON DIFFERENTIAL GAMES

The solution concepts of N-person differential games should essentially be the same as those discussed in Section 2.2, since these concepts are defined for the Normal Form of the game which does not explicitly include the specific constraints in the game. Thus the solution of the game itself depends upon the information patterns to the players and other constraints on communication and cooperation between them. The actual normal form of a differential game will be obtained in Chapter III, but we discuss the implications of the concepts here.

The game formulated in Section 2.3 in (2.11)-(2.18) is said to have an Equilibrium Point if strategies $(U^{1*}, \ldots U^{N*})$ exist such that the following holds for $p = 1, \ldots N$.

$$J^p[x_o, t_o, \underline{U}^*] \leq J^p[x_o, t_o, (\underline{U}^*; U^p)] \qquad (2.21)$$

The strategies $\underline{U}^*$ and $(\underline{U}^*; U^p)$ are further restricted to be Playable which means that they assure Termination of the game which is necessary for the evaluation of the various payoff functionals.

The strategy of any player is to be remembered as a function of the information to him. Thus under perfect

information a player employs a Closed-loop control law as his strategy and under no observations (except the initial conditions), he has to implement his strategy as an Open-loop control law. Considering a finite multistage game, Starr and Ho (1969 b) illustrated specifically that the Nash equilibrium solutions are different with these two different information patterns, which are the ones considered in detail in the following.

The question of whether the Principle of Optimality applying in some form to the Nash equilibrium solution has been raised by Starr and Ho (1969 b). It is obvious that the Principle of Optimality, that any part of the optimal solution or trajectory is optimal between its end points, applies equally well here. For players having perfect information and hence use closed-loop control laws, the Imbedding Principle is valid and along with the Principle of Optimality yields direct from the definition of the equilibrium point,

$$J^p[x,\ t,\ \underline{U}^*] \leqslant J^p[x,\ t,\ (\underline{U}^*;\ U^p)] \qquad (2.22)$$

for any x, t by the Dynamic Programming argument. On the other hand, for players having no observations, the question of multiple moves or stages is only artificial and illusory at least to the players concerned and the dynamic programming argument does not arise. The implications of these

remarks on the necessary conditions satisfied by these strategies will be examined now.

The Minimum Principle, as is originally stated by Pontryagin, can be viewed as the outgrowth of the Hamiltonian approach to variational problems and is applicable to open-loop control laws. The generalization of this to the Nash equilibrium situation as stated by Karvovskiy and Kuznetsov (1966) and Case (1967) is thus applicable to games with no observations to the players except the initial conditions. In contrast to this, Dynamic Programming method is a Value-Function approach similar to the Hamilton-Jacobi theory and is applicable to closed-loop control laws. The generalization of this as stated by Sarma et. al. (1969) and Starr and Ho (1969 a) applies to games with perfect information to the players. The derivation of these results will be persued in Chapter III.

The inequality in (2.21) and (2.22) requires a simultaneous selection of strategies by the players. Variation of the equilibrium concept with a Heirarchical Type of Information similar to the Theory of Minimax (Danskin 1967) appears in differential games studied by Soviet Authors (for example Pontryagin 1966 and Gindes 1967). According to this, the N^{th} player chooses his strategy first, then knowing this the $(N-1)^{st}$ player and so on down the line and finally the first player with the knowledge of all

the other strategies, all under noncooperation. The inequality (2.21) gets modified as follows:

$$
\begin{aligned}
& J^1[x_o, t_o, u^1, u^2, .. u^N] \geqslant J^1[x_o, t_o, u^{1*}, u^2, .. u^N] \\
& J^2[x_o, t_o, u^{1*}, u^2, .. u^N] \geqslant J^2[x_o, t_o, u^{1*}, u^{2*}, .. u^N] \\
& \vdots \\
& J^N[x_o, t_o, u^{1*}, u^{2*} .. u^N] \geqslant J^N[x_o, t_o, u^{1*}, u^{2*} .. u^{N*}]
\end{aligned}
\tag{2.23}
$$

The inequality (2.22) also gets modified similarly under the same assumptions about the players' observations of the state of the game.

A control action $\underline{u}^o$ is said to be Pareto Optimal if for any other control action $\underline{u}$, the following is true.

$$
\left\{ \begin{matrix} J^p[x_o, t_o, \underline{u}] \leqslant J^p[x_o, t_o, \underline{u}^o] \\ p = 1, \dots N \end{matrix} \right\} \text{ only if }
$$

$$
\left\{ \begin{matrix} J^p|x_o, t_o, \underline{u}| = J^p|x_o, t_o, \underline{u}^o| \\ p = 1, \dots N \end{matrix} \right\} \tag{2.24}
$$

Dynamic Programming can be used successfully for obtaining $\underline{u}^o$ if such control actions are finite in number (see Zadeh 1963). In continuous-time deterministic problems, this is not true and necessary conditions similar to Pontryagin's minimum principle are used for the purpose (see Chapter V). Since there is perfect agreement in the beginning between the players, it can be implemented in closed-loop or open-loop depending upon the players' observations.

Similar to finite games, we assume that the players strike cooperation at the Commencement of the Play and decide on a Pareto Optimal strategy. In implementing this strategy individually, each player should have confidence in the others' adherence to the agreed strategy. If the player has perfect information, his running knowledge of state enables him to detect about the departure of other players, the matter being particularly simple in the case of two-player Games.

We will not consider the delaying tactics, if any, of the players in postponing cooperation to a later stage (see for example Lawser and Volz 1969). We believe that such can be discussed in a model which allows the Cooperation and Communication between the players as moves in the model.

2.5 CONCLUSIONS

A general class of N-person Differential Games are formulated in this chapter. The solution concepts of finite games seem applicable to Differential Games as well. The concepts of closed-loop and open-loop control laws in control theory are applicable mainly to Two Different Information Patterns, perfect and null information respectively to the players. The two main approaches for .ving the necessary conditions - Pontryagin's minimum

principle and Dynamic Programming - are thus applicable to these cases respectively. This is the subject matter of the next chapter.

We restrict our attention to Pure Strategies only in this thesis because of the difficulties in implementing mixed strategies. Behaviour strategies are easier to implement, being associated with the information sets and we feel they have a lot of importance in Stochastic Differential Games.

CHAPTER III

NECESSARY CONDITIONS FOR NONCOOPERATIVE SOLUTION

3.1 INTRODUCTION

In the last chapter we saw that the concept of Nash Equilibrium is central to the noncooperative solution of a game. In general, for any problem which is fairly complex, one has to resort to the application of suitable Necessary Conditions for the determination of the equilibrium strategies. Alternate approaches are possible for simpler problems. For example, Petrosyan (1965) reduces a multi-pursuer multi-evader game under noncooperation into an Integer Programming Problem by considering the component games with multiple pursuers and single evader and single pursuer and multiple evaders.

In this chapter, we shall obtain a Modified Minimum Principle for the equilibrium strategies by the application of Dynamic Programming, a rigorous version of which appears in (Sarma et. al. 1969). For the class of games studied in this chapter, we assume Complete and Perfect Information to the players and the existence of a unique equilibrium point in pure strategies over the region of interest except for starting points whose 'measure is zero'. Surfaces containing Abnormal and

Singular Solutions and other switching surfaces will be studied in the next chapter.

As it has become customary to study linear problems with quadratic cost functionals (see for example Bellman 1967, Ho et. al. 1965, Starr and Ho 1969 a and Rhodes 1969), we shall consider one such problem. Such problems provide good insight and are amenable for analytical manipulations. The second example we consider is the noncooperative solution of a Double Integral Plant with time and fuel minimization.

3.2 NECESSARY CONDITIONS FOR NONCOOPERATIVE SOLUTION

Before proceeding to the derivation of the necessary conditions, we construct the Normal Form of the deterministic N-person differential game with perfect information to all the players formulated in Section 2.3. The relevant equations are reproduced here since frequent reference is made to them in this chapter.

The game satisfies the state equation

$$\dot{x} = f(x, \underline{u}, t) \tag{3.1}$$[1]

where x and u^p are of dimensions n and r^p respectively. The region of interest in the state-time space in which the game takes place is $\mathcal{R}$, known as the Playing

1 We use the underbar notation when similar quantities or variables related to all the players are put together. Thus $\underline{u} = (u^1, .. u^p, .. u^N)$.

Space and the terminal surface $\mathcal{T}$ is part of its boundary made up of the union of smooth surfaces

$$= \bigcup_{i=1}^{a} \mathcal{T}_i \tag{3.2}$$

Each $\mathcal{T}_i$ is given by the equations

$$t_f = T_{ij_i}(\sigma) \quad ; \quad x_f = X_{ij_i}(\sigma) \qquad (3.3)^2$$

The players choose strategies as functions of state and time satisfying the control variable constraints, i.e.,

$$u^p = U^p(x, t) \tag{3.4}$$

such that

$$u^p \in \Omega^p(x, t) \tag{3.5}$$

where Ω^p is a restraint set given by an inequality of dimension l^p such as,

$$K^p(x, u^p, t) \quad 0 \tag{3.6}$$

The p^{th} player minimizes his payoff functional

$$J^p[\xi, \tau, \underline{U}] = \phi^p(x_f, t_f) + \int_{\tau}^{t_f} L^p(x, \underline{U}(x,t), t)\, dt \qquad (3.7)^3$$

We assume that the functions L^p, f and K^p and their partial derivatives with respect to x are continuous in their arguments, i.e., they are of class $C^{(1)}$. Similarly each T_{ij_i} and X_{ij_i} and the function ϕ^p are of class $C^{(1)}$ on each $\mathcal{T}_i$.

---

2 The subscript j_i is associated with a Regular Decomposition to be introduced below.

3 Here ϕ^p can be expressed as a function of σ in view of (3.3).

Normal Form of the Game:

Let Σ^p be the class of all functions satisfying (3.4)-(3.6) which are piecewise continuous with piecewise continuous derivatives, i.e., they belong to the class piecewise $C^{(1)}$. For any $U^p \in \Sigma^p$, the solution to (3.1) called paths may Bifurcate or Coalesce at the points of discontinuity of one or more U^p. Otherwise the solution will be unique.

We shall say that $U^p \in \Sigma^p$, $p = 1, \ldots N$, or $\underline{U} \in \underline{\Sigma}$ is a Playable N-tuple if for each $(\xi, \tau) \in \mathcal{R}$, every solution stays in $\mathcal{R}$ and reaches the terminal surface in finite time. Thus playability is Joint Controllability of the state of the game and the payoff J^p can be multi-valued because of the bifurcation in paths cited earlier. We consider maximum nonvoid subclasses $\mathcal{U}^p \subseteq \Sigma^p$, such that $U^p \in \mathcal{U}^p$, $p = 1, \ldots N$ is a playable N-tuple of strategies. Thus A Normal Form of the game is given by

$$\left\{\mathcal{U}^1, \ldots \mathcal{U}^N ; J^1, \ldots J^N\right\} \tag{3.8}$$

with $\mathcal{U}^1, \ldots \mathcal{U}^N$ as the pure strategies of the respective players.

Assumptions on Optimal Paths:

Let $\underline{U}^*$ be the noncooperative solution in terms of equilibrium points for the game (3.8). For the class of games considered here $\underline{U}^*$ exists. Thus if there is more

than one path starting from a point $(\xi, \tau) \in \mathcal{R}$ because of discontinuity in the strategies of some players, then the payoffs to These Players are independent of the various paths and by assumption, we have

$$J^p[\xi, \tau, (\underline{U}^* ; U^p)] \geqslant J^p[\xi, \tau, \underline{U}^*] = W^p(\xi, \tau) \tag{3.9}$$

where $\underline{W}(\xi, \tau)$ is called the Value Function of the noncooperative game.

We make the following assumptions on $\underline{U}^*$ and the associated solutions x^* to (3.1) called the optimal paths.

(i) $\underline{U}^*$ exists

(ii) The decomposition associated with $\underline{U}^*$ is Regular (see Figure 3.1). This consists of disjoint open subregions $\mathcal{R}_{ij}$, $j = 1, \ldots j_i$ and the switching surfaces $\mathcal{M}_{ij}$ and $\mathcal{N}_{i_1, \ldots i_k}$.

The manifold $\mathcal{M}_{ij}$ separates the subregions $\mathcal{R}_{ij}$ and $\mathcal{R}_{i,j+1}$. Although the manifold arises out of the use of discontinuous strategies by some of the players, the Value Function $\underline{W}$ is continuous across $\mathcal{M}_{ij}$. The manifold $\mathcal{M}_{ij}$ can be expressed as

$$t = T_{ij}(\sigma) \quad ; \quad x = X_{ij}(\sigma) \tag{3.10}$$

where σ ranges over an n-dimensional cube. The union of $\mathcal{M}_{ij_i}$ for all i forms the Terminal Surface $\mathcal{J}$ given by (3.2) and (3.3) (see Footnote 2).

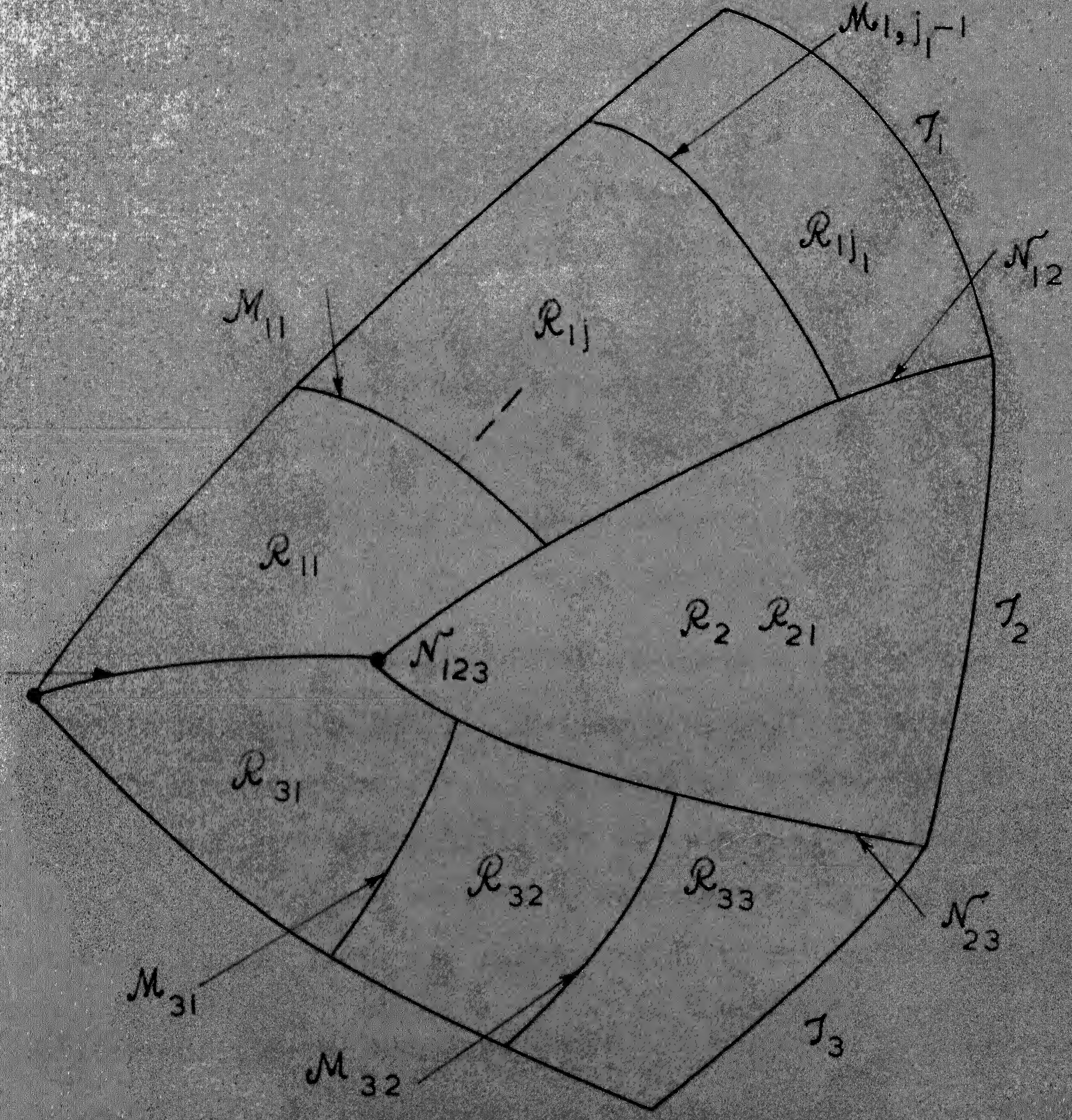

FIG. 3.1 REGULAR DECOMPOSITION OF THE PLAYING SPACE $\mathcal{R}$.

The manifold $\mathcal{N}_{i_1, \ldots i_k}$ is the intersection of the subregions $\mathcal{R}_{i_1}, \ldots \mathcal{R}_{i_k}$. For starting points on this manifold there may be multiple optimal paths arising out of the discontinuities in the strategies of some players. The Value Functions of Only These Players[4] are continuous across the manifold.

(iii) If the initial point is interior to one of the subregions $\mathcal{R}_{ij}$ then the optimal trajectory x^* is unique.

(iv) The optimal paths are never tangential to any of the switching manifolds or the terminal surface.

Further properties of the optimal paths x^* are given in (Berkovitz 1967). The above assumptions together with the n-dimensionality of the terminal surface make the paths Normal (Berkovitz 1961) and the Abnormal and Singular surfaces and surfaces containing Perpetuated Dilemma to the players (Isaacs 1965) are ruled out in the present context.

Hamilton-Jacobi-Bellman Equations:

We shall now derive the Hamilton-Jacobi-Bellman equations satisfied by the Value Function $\underline{W}$ by the application of Dynamic Programming. For this, the game is considered as viewed by each of the players when all he

4 The Value Functions of the players using continuous strategies need not be continuous. This arises out of the nonequivalence of the equilibrium points and will be pursued in Chapter IV.

knows is that the rest have perhaps chosen their noncooperative optimal strategies.

Let (ξ, τ) be a point of $\mathcal{R}_{ij}$. Then for the p^{th} player,

$$W^p(\xi, \tau) = \phi^p(x_f, t_f) + \left\{ \int_{\tau}^{t_{ij}} + \sum_{k=j}^{j_i-1} \int_{t_{ik}}^{t_{i,k+1}} \right\} L^{p*} \, dt \quad (3.11)$$

where

$$L^{p*} = L^p(x^*, \underline{U}^*(x^*, t), t) \quad (3.12)$$

Thus from the properties of the optimal paths (Berkovitz 1967), it follows that W^p_ξ, W^p_τ[5] exist and are continuous on $\mathcal{R}_{ij}$ with unique one-sided limits onto $\mathcal{M}_{i,j-1}$ and $\mathcal{M}_{ij}$.

To establish the partial differential equation satisfied by $W^p(\xi, \tau)$, we consider the particular nonoptimal strategy for the p^{th} player,

$$\hat{U}^p(x,t) = \begin{cases} U^p(x,t) & , \quad (x,t) \in N(\xi, \tau) \\ U^{p*}(x,t) & , \quad (x,t) \notin N(\xi, \tau) \end{cases} \quad (3.13)$$

where $U^p \in \mathcal{U}^p$ and $N(\xi, \tau)$ is a neighbourhood of (ξ, τ) wholly contained in $\mathcal{R}_{ij}$. It can be shown that $\hat{U}^p \in \mathcal{U}^p$ and let δ stand for the last time the trajectory leaves $N(\xi, \tau)$.

Now since $\underline{U}^*$ is equilibrium optimal

$$W^p(\xi, \tau) = J^p(\xi, \tau, \underline{U}^*) \leq J^p(\xi, \tau, (\underline{U}^* ; \hat{U}^p)) \quad (3.14)$$

---

5 Variable subscripts indicate partial derivatives in the following.

The right-hand side of the inequality (3.14) can be expanded as

$$\phi^p(x_f, t_f) + \left(\int_{\tau}^{\tau+\delta} + \int_{\tau+\delta}^{t_f} \right) L(x, (\underline{U}^* ; \hat{U}^p), t)\, dt$$

$$= \int_{\tau}^{\tau+\delta} L^p(x, (\underline{U}^* ; \hat{U}^p), t)\, dt + W^p(x(\tau + \delta), \tau + \delta) \quad (3.15)$$

Hence in view of (3.13), we have

$$-W^p(x(\tau+\delta), \tau+\delta) + W^p(\xi, \tau) \leq \int_{\tau}^{\tau+\delta} L^p(x, (\underline{U}^*; U^p), t)\, dt \quad (3.16)$$

with the equality holding for $U^p = U^{p*}$.

We shall now let $N(\xi, \tau) \to (\xi, \tau)$ which implies that $\delta \to 0$. Since W^p_ξ and W^p_τ are continuous on $\mathcal{R}_{ij}$, we can apply the mean value theorem to the inequality (3.14) and write the left-hand side as

$$-W^p_\tau(\xi, \tau)\delta - W^p_\xi(\xi, \tau)[x(\tau+\delta) - \xi] + o(\delta)$$

$$= -W^p_\tau(\xi, \tau)\delta - W^p_\xi(\xi, \tau) f(\xi, (\underline{U}^*(\xi, \tau); U^p(\xi, \tau)), \tau)\delta + o(\delta) \quad (3.17)$$

Similarly the right-hand side can be written as $L^p(\xi, (\underline{U}^*(\xi, \tau); U^p(\xi, \tau)), \tau)\delta + o(\delta)$ and in the limit the inequality (3.16) reduces to

$$-W^p_\tau(\xi, \tau) \leq L^p(\xi, (\underline{U}^*(\xi, \tau); U^p(\xi, \tau)), \tau) + W^p_\xi(\xi, \tau) f(\xi, (\underline{U}^*(\xi, \tau); U^p(\xi, \tau)), \tau) \quad (3.18)$$

Since U^p is arbitrary, (3.18) holds for $U^p \in \mathcal{U}^p$ with the equality holding for $U^p = U^{p*}$. It can also be

written in the following convenient forms.

$$
\begin{aligned}
-W^p_\tau(\xi,\tau) &= L^p(\xi,\underline{U}^*(\xi,\tau),\tau) + W^p_\xi(\xi,\tau)f(\xi,\underline{U}^*(\xi,\tau),\tau) \\
&= H^p(\xi,W^p_\xi(\xi,\tau),\ \underline{U}^*(\xi,\tau),\tau) \\
&= \min_{U^p \in \mathcal{U}^p} H^p(\xi,W^p_\xi(\xi,\tau),(\underline{U}^*(\xi,\tau);U^p(\xi,\tau)),\tau) \qquad (3.19) \\
&= \min_{U^p \in \mathcal{U}^p} L^p(\xi,(\underline{U}^*(\xi,\tau);\ U^p(\xi,\tau)),\tau) \\
&\qquad + W^p_\xi(\xi,\tau)f(\xi,(\underline{U}^*(\xi,\tau);U^p(\xi,\tau)),\tau)
\end{aligned}
$$

where the Hamiltonian Function H^p is defined as

$$
H^p(x,\lambda,u,t) = \lambda^p_o\ L^p(x,u,t) + \lambda^p \cdot f(x,u,t) \qquad (3.20)
$$

In this chapter, it is invariably assumed that λ^p_o is equal to unity[6].

Equation (3.19) is termed the Hamilton-Jacobi-Bellman equation and holds for $p = 1, \ldots N$.

The Minimum Principle for the Players:

The necessary conditions obtained above can be expressed in the Hamiltonian Form by introducing adjoint variables or Lagrange multipliers and relating them to W^p_t and W^p_x.

Let (ξ,τ) be a point in $\mathcal{R}_{ij}$. We consider the following linear differential equation with the final condition $\lambda^p(t_f) = \lambda^p(\tau_{ij_1}(\cdot)) = \lambda^p_{j_1}$:

6 Abnormal paths, on which $\lambda^p_o = 0$, will be discussed in the next chapter.

$$\dot{\lambda}^p = -\left(H_x^{p*} + \sum_{\ell} H_{u^\ell}^{p*} U_x^{\ell *} \right) \tag{3.21}$$

where H^p is given by (3.20) and the star notation is used as in (3.12) to indicate that the arguments are in terms of variables related to the optimal paths. Also let the components of $\lambda_{j_1}^p$ be given by the following system of linear equations.

$$\lambda_o^p[\phi_\sigma^p + L^p(\pi_{ij_1}) \frac{\partial T_{ij_1}}{\partial \sigma}] + \lambda_{j_1}^p [f(\pi_{ij_1}) \frac{\partial T_{ij_1}}{\partial \sigma} - \frac{\partial X_{ij_1}}{\partial \sigma}] = 0 \tag{3.22}$$

where π_{ij_1} indicates that the arguments correspond to the terminal surface. Equation (3.21) defines $\lambda_{j_1}^p$ as continuous function of (ξ,τ) on $\mathcal{R}_{ij}$ and the solution λ^p to (3.21) is also a continuous function of ξ,τ and t on $\mathcal{R}_{ij}$ by standard theorems in differential equations.

Now we define the corner conditions which determine the left-hand limits λ^{p^-} from the right-hand limits λ^{p^+} at the $\mathcal{M}_{ik}$ manifolds.

$$-\lambda_k^{p^-}[f(\pi_{ik}^-) \frac{\partial T_{ik}}{\partial \sigma} - \frac{\partial X_{ik}}{\partial \sigma}] = [\lambda_o^{p^-} L^p(\pi_{ik}^-) - \lambda_o^{p^+} L^p(\pi_{ik}^+)] - \lambda_k^{p^+}[f(\pi_{ik}^+) \frac{\partial T_{ik}}{\partial \sigma} - \frac{\partial X_{ik}}{\partial \sigma}] \tag{3.23}$$

where $\lambda_o^{p^-} = \lambda_o^{p^+} = 1$ and π_{ik}^- and π_{ik}^+ indicate that the arguments of the functions are appropriate one-sided limits at $\mathcal{M}_{ik}$.

Thus the solution λ^p of (3.21) is defined and continuous for (ξ,τ) in $\mathcal{R}_{ij}$ with unique one-sided limits on $\mathcal{M}_{ik}$ and satisfies the transversality and corner conditions (3.22) and (3.23), which can be stated in a more compact form as follows :

$$\lambda_o^p \, \phi_\sigma^p + H^p(\pi_{ij_i}) \frac{\partial T_{ij_i}}{\partial \sigma} - \lambda_{j_i}^p \frac{\partial X_{ij_i}}{\partial \sigma} = 0 \qquad (3.24)$$

$$[H^p(\pi_{ik}^+) - H^p(\pi_{ik}^-)] \frac{\partial T_{ik}}{\partial \sigma} - (\lambda^{p^+} - \lambda^{p^-}) \frac{\partial X_{ik}}{\partial \sigma} = 0 \qquad (3.25)$$

Now for the Value Function W^p, we can write for any (ξ,τ) in $\mathcal{R}_{ij}$

$$W_\xi^p(\xi,\tau) = \phi_\sigma^p \frac{\partial \sigma}{\partial \xi} + L^p(\pi_{ij_i}) \frac{\partial t_{ij_i}}{\partial \xi} + \sum_{k=1}^{j_i-1} [L^p(\pi_{ik}^-) - L^p(\pi_{ik}^+)] \frac{\partial t_{ik}}{\partial \xi}$$
$$+ \left(\int_\tau^{t_{ij}} + \sum_{k=j}^{j_i-1} \int_{t_{ik}}^{t_{i,k+1}}\right)(L_x^{p*} + \sum_\ell L_{u^\ell}^{p*} U_x^{\ell*}) x_\xi^* dt \qquad (3.26)$$

The terms involving integrals can be rewritten using (3.21) as follows :

$$\left(\int_\tau^{t_{ik}} + \sum_{k=j}^{j_i-1} \int_{t_{ik}}^{t_{i,k+1}}\right)(-\dot{\lambda}^p - \lambda^p f_x^* - \sum_\ell \lambda^p f_{u^\ell}^* U_x^{\ell*}) x_\xi^* dt \qquad (3.27)$$

Further, in view of the system equation (3.1), on the optimal path x^*, we have

$$\dot{x}_\xi^* = (f_x^* + \sum_\ell f_{u^\ell}^* U_x^{\ell*}) x_\xi^* \qquad (3.28)$$

and (3.26) can be simplified as

$$-\left(\int_{\tau}^{t_{ij}} + \sum_{k=j}^{j_i-1} \int_{t_{ik}}^{t_{i,k+1}}\right) d(\lambda^p x^*)$$

$$= -\lambda^p_{j_i} x^*_{\xi}(t_{ij_i}) + \lambda^p(\xi,\tau,\tau)$$

$$+ \sum_{k=j}^{j_i-1} [\lambda^{p+}_k x^*_{\xi}(t_{ik}+) - \lambda^{p-}_k x^*_{\xi}(t_{ik}-)] \qquad (3.29)$$

Now, from (3.29), (3.26), (3.22) and (3.23), it follows that

$$W^p_{\xi}(\xi,\tau) = \lambda^p(\xi,\tau,\tau) \qquad (3.30)$$

and

$$W^p_x(x,t) = \lambda^p(\xi,\tau,t) = \lambda^p(x,t,t) \qquad (3.31)$$

Thus we can write (3.19) as follows:

$$-\ W^p_x(x,t) = H^p(x^*,\lambda^p,\underline{u}^*,t)$$

$$= \min_{u^p} H^p(x^*,\lambda^p,(\underline{u}^*;u^p),t) \qquad (3.32)$$

where $u^p = U^p(x^*,t)$ for some $U^p \in \mathcal{U}^p$. Equation (3.32) holds for all the players, i.e., $p = 1, \ldots N$ and implies that at any point (x,t) in $\mathcal{R}$ the game $\mathcal{G}(x,t)$ with payoffs defined by $H^p(x,\lambda^p,\underline{u},t)$ has a pure strategy equilibrium point $\underline{u}^*$. The value of the game is

$$[\ H^1(x,\ ^1,\underline{u}^*,t),\ \ldots\ H^N(x,\ ^N,\underline{u}^*,t)\ ] = -\underline{W}_t(x,t) \qquad (3.33)$$

Let the function $K^p(x,u^p,t)$ in (3.6) which define the restraint sets $\Lambda^p(x,t)$ satisfy the constraint

conditions that if $l^p \geq r^p$, then at each point (x, u^p, t) at most r^p components of K^p can vanish and the matrix $\left| \frac{\partial \hat{K}^p}{\partial u^p} \right|$ formed from $\hat{K}^p$, the vanishing components of K^p, has maximum rank, this being true for $p = 1, \ldots N$. Then there exist functions μ^p such that the following hold (Berkovitz 1967).

$$H^p_{u^p} + \mu^p \cdot K^p_{u^p} = 0 \tag{3.34}$$

$$\mu^p \leq 0 \tag{3.35}$$

$$\mu^p \cdot K^p = 0 \tag{3.36}$$

Thus when u^p is interior to its restraint set Ω^p, i.e., $K^p > 0$, (3.36) yields that $\mu^p = 0$. Thus condition (3.34) reduces to an important form

$$H^p_{u^p} = 0 \tag{3.37}$$

Equation (3.37) is also true when u^p is unconstrained, i.e., when (3.6) is absent.

3.3 FURTHER RESULTS PERTAINING TO THE EQUILIBRIUM CONCEPT

We obtained in Section 3.2 the necessary conditions for equilibrium strategies based on the results of Berkovitz (1967). Alternately, one can pose an optimal control problem for each player against the equilibrium play of the rest of the players. The equivalence of the necessary conditions

for all these problems and those for the equilibrium point of the normal form of the game as constructed in Section 3.2, is shown by Berkovitz (1964). The following results are stated keeping this equivalence in mind.

Legendre-Clebsch Condition:

At any point on an optimal path excluding corners, if $\hat{K}^p$ is the vector formed from K^p by taking those components that vanish at that point, then for all e^p satisfying $\hat{K}^p_{u^p} \cdot e^p = 0$, it follows that

$$e^p((H^p + \mu^p K^p)_{u^p u^p}) e^p \geqslant 0 \tag{3.38}$$

If u^p is interior to $\mathcal{A}^p$ or u^p is unconstrained, then μ^p becomes a null vector by (3.36) and (3.38) reduces to the easier and classical form viz. for all e^p,

$$e^p(H^p_{u^p u^p}) e^p \geqslant 0 \tag{3.39}$$

or that $H^p_{u^p u^p}$ is positive semidefinite.

If on an extremal trajectory excluding corners, (3.39) is satisfied with strict inequality, i.e., $H^p_{u^p u^p}$ is positive definite, then the Legendre-Clebsch condition is said to be satisfied in the strengthened form. If on the other hand, $H^p_{u^p u^p}$ is positive semidefinite, the presence of Singular Control Variables is indicated and these are discussed in the next chapter.

Null-Observation Games:

We observed in Chapter II, that a player p with no observations implements his strategy U^p as an open-loop law, i.e., his control action is given by

$$u^p = U^p(x_o, t_o, t) \tag{3.40}$$

Though there is no use defining a value function for this player, Pontryagin's minimum principle holds for his one-sided optimal control problem. Thus the necessary conditions of Section 3.2 hold with the difference that U^p_x will be zero. Therefore if none of the players have any observations, the adjoint equations (3.21) will have the form (Case 1967 and Starr and Ho 1969)

$$\lambda^p = - \frac{\partial H^p}{\partial x} \tag{3.41}$$

Powerful mathematical tools like Functional Analysis which are mainly applicable to open-loop controls (Gindes 1967 and Kirillova 1967) are applicable to some of the problems of this category.

Sufficient Conditions:

The sufficiency conditions in the literature of optimal control and variational calculus are based either on the Value Function Approach or on the Conjugate Point Condition. The Value Function method is primarily

applicable for the closed-loop control laws and the conjugate point method for the open-loop control laws. Thus we state two simple sufficient conditions below.

For the class of perfect information games, studied in this chapter, i.e., having no Abnormal and Singular solutions, if the strategies satisfy the following Hamilton-Jacobi equations with their resulting Values, then the strategies are optimal.

$$- W_t^p(x,t) = \min_{u^p \in \Omega^p} H(x, W_x^p(x,t), (\underline{u}^*; u^p), t) \qquad (3.42)$$

It is to be noted that this condition is global and is a stronger requirement than that in (3.19) and (3.32).

If on an extremal trajectory of a null-observation game, the strengthened Legendre-Clebsch condition is satisfied, then a necessary and sufficient condition that any player's strategy is optimal is that there be no conjugate points for the Accessory Minimization Problem of this player which will be a linear-quadratic-optimal-control problem. From the results in (Breakwell and Ho 1965 and Schmitendorf and Citron 1969), it follows that the Riccati equations of the players in the Accessory Game (which is linear-quadratic once again) should have bounded solutions.

3.4 EXAMPLES

We consider two examples in this section to illustrate the application of the results in Sections 3.2 and 3.3. The first is a game with linear dynamics and

quadratic payoff functionals studied by Starr and Ho (1969) and Rhodes (1969) in which the optimal strategies are continuous. The second example involves time and fuel minimization of a double integral plant emphasizing the presence of discontinuous optimal strategies. We dwell on this example at considerable length in the later chapters.

Example 3.1 :

The state of the game, x of dimension n, satisfies the linear differential equation

$$\dot{x} = A(t)\, x(t) + \sum_{p=1}^{N} B^p(t)\, u^p(t) \tag{3.43}$$

where u^p, the unconstrained control action vector of the player p, is of dimension r^p. The matrices $A(t)$ and $B^p(t)$ are of dimensions $n \times n$ and $n \times r^p$ respectively.

The payoff functional of the p^{th} player is given by

$$J^p|x_o, t_o, \underline{u}| = x_f^T F^p x_f + \int_{t_o}^{t_f} |x^T(t) Q^p(t) x(t) + \sum_{j=1}^{N} u^{j\,T}(t) R_j^p(t) u^j(t)|\, dt \tag{3.44}$$ [7]

where x_o is the initial state at time t_o and the final state x_f, at the specified t_f, is arbitrary. The matrices F^p, $Q^p(t)$ and $R_j^p(t)$ for $p, j = 1, \ldots N$ are

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7 Transpose Notation is used only in this example because of the familiarity of the results for these problems in this form.

symmetric matrices of proper dimension. The matrix $R_p^p(t)$ is assumed to be positive definite.

The game with perfect information to all the players can be solved by the Value Function approach. Assuming $W^p(x,t)$ to be of the form

$$W^p(x,t) = x^T(t)\, S^p(t)\, x(t) \tag{3.45}$$

we have from the Hamilton-Jacobi equation in any of the forms (3.19), (3.32) or (3.42),

$$- x^T(t)\dot{S}^p(t)x(t) = \min_{u^p}\Big[2\langle S^p(t)x(t), A(t)x(t) + \sum_j B^j(t)u^j(t)\rangle + x^T(t)Q^p(t)x(t) + \sum_j u^{j^T}(t)R_j^p(t)u^j(t)\Big] \tag{3.46}$$

or solving (3.46)

$$u^p(t) = -R_p^{p^{-1}}(t)\, B^{p^T}(t)\, S^p(t)\, x(t) \tag{3.47}$$

Substituting (3.47) into (3.46), we have after dropping the argument t,

$$\dot{S}^p = -S^pA - A^TS^p - Q^p - \sum_j (S^jB^jR_j^{j^{-1}}R_j^pR_j^{j^{-1}}B^{j^T}S^j - S^pB^jR_j^{j^{-1}}B^{j^T}S^j - S^jB^jR_j^{j^{-1}}B^{j^T}S^p) \tag{3.48}$$

with the final condition

$$S^p(t_f) = F^p \tag{3.49}$$

Equations (3.47)-(3.49) for $p = 1, \dots N$ is the noncooperative solution of the game.

The game with no observations to all the players can be solved by posing the equivalent tracking problems to the various players as follows. We define x^o, x^p, ϱ^p and ξ^p for $p = 1, \dots N$ as

$$\dot{x}^o = A(t)\, x^o(t) \quad ; \quad x^o(t_o) = x_o \tag{3.50}$$

$$\dot{x}^p = A(t)\, x^p(t) + B^p(t)\, u^p(t) \quad ; \quad x^p(t_o) = 0 \tag{3.51}$$

$$\varrho^p = x^o + x^p \tag{3.52}$$

$$\xi^p = - \sum_{j \neq p} x^j \tag{3.53}$$

Now it is obvious that

$$x = x^o + x^1 + \dots x^N = \varrho^p - \xi^p \tag{3.54}$$

For the p^{th} player, thus we have to minimize

$$J^p[x_o, t_o, (\underline{u}; u^p)] = \| \varrho^p_f - \xi^p_f \|^2_{F^p} + \int_{t_o}^{t_f} \| \varrho^p - \xi^p \|^2_{Q^p} + \| u^p \|^2_{R^p_p} + \sum_{j \neq p} \| u^j \|^2_{R^p_j} \, dt \tag{3.55}$$ [8]

Since the last term in the integral of (3.55) is outside the choice of the p^{th} player, the solution to the above problem can be obtained as the solution of the following tracking problem.

Determine $u^p(t)$ to minimize

$$J^p[\varrho^p_o, t_o, u^p] = \| \varrho^p_f - \xi^p_f \|^2_{F^p} + \int_{t_o}^{t_f} \| \varrho^p(t) - \xi^p(t) \|^2_{Q^p(t)} + \| u^p(t) \|^2_{R^p_p(t)} \, dt \tag{3.56}$$

[8] For convenience Norm notation is used here to indicate the Quadratic Forms.

Example 3.2

The state of the game satisfies the differential equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u^1 + c\, u^2 \end{aligned} \tag{3.62}$$

where the control variables of the two players, u^1 and u^2 respectively are constrained as follows:

$$|u^1| \leq 1 \quad ; \quad |u^2| \leq 1 \tag{3.63}$$

The players 1 and 2 choose their control action so as to minimize respectively their payoff functionals

$$\begin{aligned} J^1[x_o,u^1,u^2] &= \int_0^{t_f} dt \\ J^2[x_o,u^1,u^2] &= \int_0^{t_f} [\, |u^1| + b|u^2| \,]\, dt \end{aligned} \tag{3.64}$$

while driving the state of the game to the origin from an arbitrary initial point x_o in the state space at time $t = 0$ to the origin, i.e., $x_1(t_f) = x_2(t_f) = 0$, where t_f is free.

The equations (3.62) will be referred to hereafter as the Double Integral Plant because of their form. The full significance of the example will be discussed in the later chapters. Here we confine ourselves to the noncooperative solution for the case $c > b$ to illustrate the theory developed in this chapter.

The application of the necessary conditions of Section 3.2 yields the following

$$H^1(x,\underline{u},\lambda^1) = 1 + \lambda_1^1 x_2 + \lambda_2^1(u^1 + cu^2)$$
$$H^2(x,\underline{u},\lambda^2) = |u^1| + b|u^2| + \lambda_1^2 x^2 + \lambda_2^2(u^1 + cu^2) \tag{3.65}$$

$$u^{1*} = - \operatorname{sgn} \lambda_2^1$$
$$u^{2*} = - \operatorname{dez}(\lambda_2^2 c/b) \tag{3.66}$$

where we define

$$\operatorname{sgn} y = \begin{cases} +1 & y > 0 \\ -1 & y < 0 \end{cases} \tag{3.67}$$

$$\operatorname{dez} y = \begin{cases} +1 & y > 1 \\ 0 & -1 < y < 1 \\ -1 & y < -1 \end{cases} \tag{3.68}$$

The adjoint equations for the perfect information case are given in terms of U^{1*}, U^{2*}, the optimal strategies of the players as

$$\dot{\lambda}_1^1 = -\lambda_2^1 c \frac{\partial U^{2*}}{\partial x_1}$$
$$\dot{\lambda}_2^1 = -\lambda_1^1 - \lambda_2^1 c \frac{\partial U^{2*}}{\partial x_2} \tag{3.69}$$

$$\dot{\lambda}_1^2 = -\lambda_2^2 \frac{\partial U^{1*}}{\partial x_1} - (\operatorname{sgn} u^1) \frac{\partial U^{1*}}{\partial x_1}$$
$$\dot{\lambda}_2^2 = -\lambda_1^2 - \lambda_2^2 \frac{\partial U^{1*}}{\partial x_2} - (\operatorname{sgn} u^1) \frac{\partial U^{1*}}{\partial x_2} \tag{3.70}$$

The terms $\frac{\partial U^{p^*}}{\partial x_j}$ for p, j = 1, 2 appearing in (3.69) and (3.70) are absent for the null observation case. However even in the perfect information case, since the value of u^{1^*} is ± 1 and that of u^{2^*} is ± 1 or 0 because of (3.66)-(3.68), the terms $\frac{\partial U^{p^*}}{\partial x_j}$ for $p,j = 1,2$ equal zero and thus the adjoint equations will be the same and are as follows for both the cases.

$$\begin{aligned} \lambda_1^p &= 0 \\ \lambda_2^p &= -\lambda_1^p \end{aligned} \tag{3.71}$$

for p = 1, 2.

We shall construct the solution by integrating backwards in time the canonical equations (3.62) and (3.71) using (3.66) starting at the terminal surface, with the transversality condition satisfied thereon. At corners, we construct the switching surfaces and continue the procedure after satisfying the corners conditions at the switching surfaces[9]. Alternatively the switching surface is assumed as a new terminal surface and the transversality condition satisfied and the procedure continued. The method is similar to that followed by Isaacs (1965).

Since the terminal specification violates the dimensionality requirement, an artifice is resorted to

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9 We name the switching surfaces obtained this way as the Transition Surfaces in the next chapter.

by choosing a new terminal surface given by

$$\begin{aligned} x_1(t_f) &= \delta \cos\theta \\ x_2(t_f) &= \delta \sin\theta \end{aligned} \tag{3.72}$$

Then we apply the transversality conditions (3.24) with $\sigma = (t_f, \theta)$

$$[\lambda_1^1(t_f) \quad \lambda_2^1(t_f)] \begin{bmatrix} \delta \sin\theta & -\delta \sin\theta \\ u^1 + cu^2 & \delta \cos\theta \end{bmatrix} = [-1 \quad 0] \tag{3.73}$$

$$[\lambda_1^2(t_f) \quad \lambda_2^2(t_f)] \begin{bmatrix} \delta \sin\theta & -\delta \sin\theta \\ u^1 + cu^2 & \delta \cos\theta \end{bmatrix} == [-\{|u^1|+b|u^2|\} \quad 0] \tag{3.74}$$

Solving (3.73) and (3.74) and letting $\delta \to 0$, we obtain

$$\lambda_1^p(t_f) = \lambda_2^p(t_f) \text{ ctn } \theta, \quad p = 1,2 \tag{3.75}$$

$$\lambda_2^2(t_f) = \{|u^1|+b|u^2|\}\lambda_2^1(t_f) = -\frac{|u^1|+b|u^2|}{u^1 + cu^2} \tag{3.76}$$

The Nash equilibrium terminal sequences consistent with (3.76), (3.71) and (3.66) are $\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} \pm 1 \\ \pm 1 \end{bmatrix}$.

For example, if we assume that

$$u^1 = u^2 = -1 \tag{3.77}$$

we have from (3.76)

$$\lambda_2^2(t_f) = \frac{(1+b)}{(1+c)} = (1+b)\lambda_2^1(t_f) \tag{3.78}$$

and since $\frac{1+b}{1+c} > \frac{b}{c}$ in view of $c > b$, we get from (3.66)

$$u^1 = -\operatorname{sgn}\lambda_2^1(t_f) = -1 \quad ; \quad u^2 = -\operatorname{dez}\frac{c\lambda_2^2(t_f)}{b} = -1 \tag{3.79}$$

Equations (3.79) and (3.77) are thus consistent.

If on the other hand, we assume that

$$u^1 = -1 \quad ; \quad u^2 = 0 \tag{3.80}$$

we get from (3.76)

$$\lambda_2^2(t_f) = \lambda_2^1(t_f) = 1 > \frac{b}{c} \tag{3.81}$$

which yields

$$\begin{aligned} u^1(t_f) &= -\operatorname{sgn}\lambda_2^1(t_f) = -1 \\ u^2(t_f) &= -\operatorname{dez}\lambda_2^2(t_f) = -1 \end{aligned} \tag{3.82}$$

Equations (3.80) and (3.82) are contradictory.

From (A.3) of Appendix A, we have the equation of the switching curve along which the state reaches origin with control law $u^1 = u^2 = -1$ as

$$\gamma_{11}^- = \left\{(s_1, s_2) : s_2 > 0 \quad , \quad s_1 = -\frac{s_2^2}{2(1+c)}\right\} \tag{3.83}$$

At this point, we can either apply the corner conditions at this switching surface (3.83) or consider γ_1^- as a new terminal surface and apply the transversality

conditions. The second method is followed here and the first method at the next corner. In essence we show that $\begin{bmatrix} +1 & +1 & -1 \\ +1 & 0 & -1 \end{bmatrix}$ is a Nash equilibrium sequence. During the course of the proof Figures 3.1 and 3.2 may be referred. Figure 3.1 shows the typical plots of λ_2^1, λ_2^2, u^1 and u^2 for this sequence and Figure 3.2 shows the switching surfaces constructed. The state of the game at times t_2 and t_3 is denoted by (r_1, r_2) and (s_1, s_2) respectively.

Since Υ_{11}^{-} is assumed as the new terminal surface, the final state is (s_1, s_2). corresponding to the new final time t_3. Now, from (A.6), we have

$$\begin{aligned} \phi^1(s_1,s_2) &= t_f - t_3 = \frac{s_2}{1+c} \\ \phi^2(s_1,s_2) &= (1+b)(t_f - t_3) = \frac{(1+b)s_2}{1+c} \\ s_1 &= -\frac{s_2^2}{2(1+c)} \end{aligned} \qquad (3.84)$$

Applying the transversality conditions (3.24) with = (s_2, t_f), we have

$$\begin{aligned} &\frac{1}{1+c} - {}^1_1\left(-\frac{s_2}{1+c}\right) - {}^1_2 = 0 \\ &1 + {}^1_1 s_2 + {}^1_2(u^1 + cu^2) = 0 \\ &\frac{1+b}{1+c} - {}^2_1\left(-\frac{s_2}{1+c}\right) - {}^2_2 = 0 \\ &|u^1| + b|u^2| + {}^2_1 s_2 + {}^2_2(u^1 + cu^2) = 0 \end{aligned} \qquad (3.85)$$

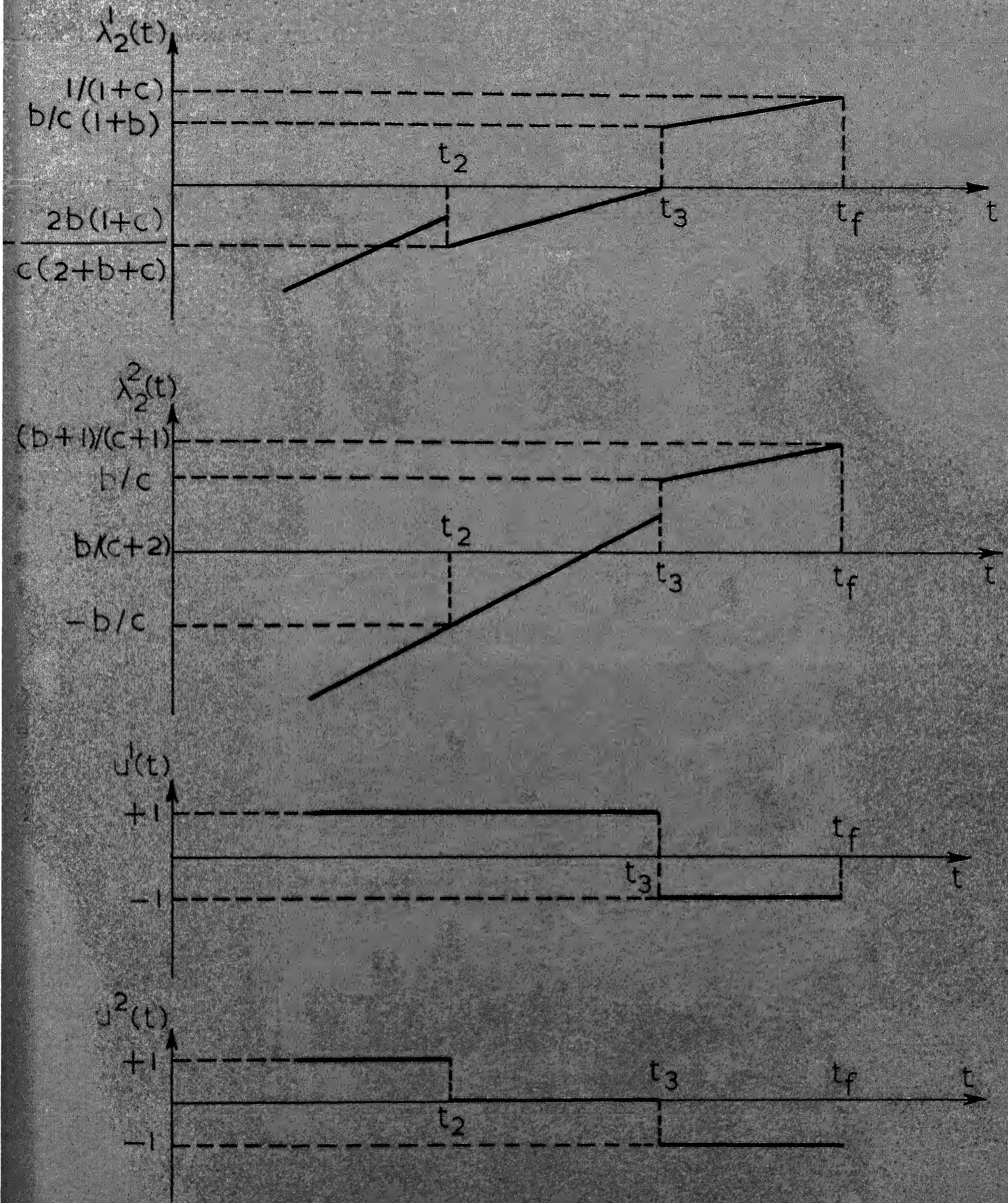


FIG. 3·2 RESPONSES OF λ AND U CORRESPONDING TO THE NONCOORERATIVE SOLUTION CASE $c > b$

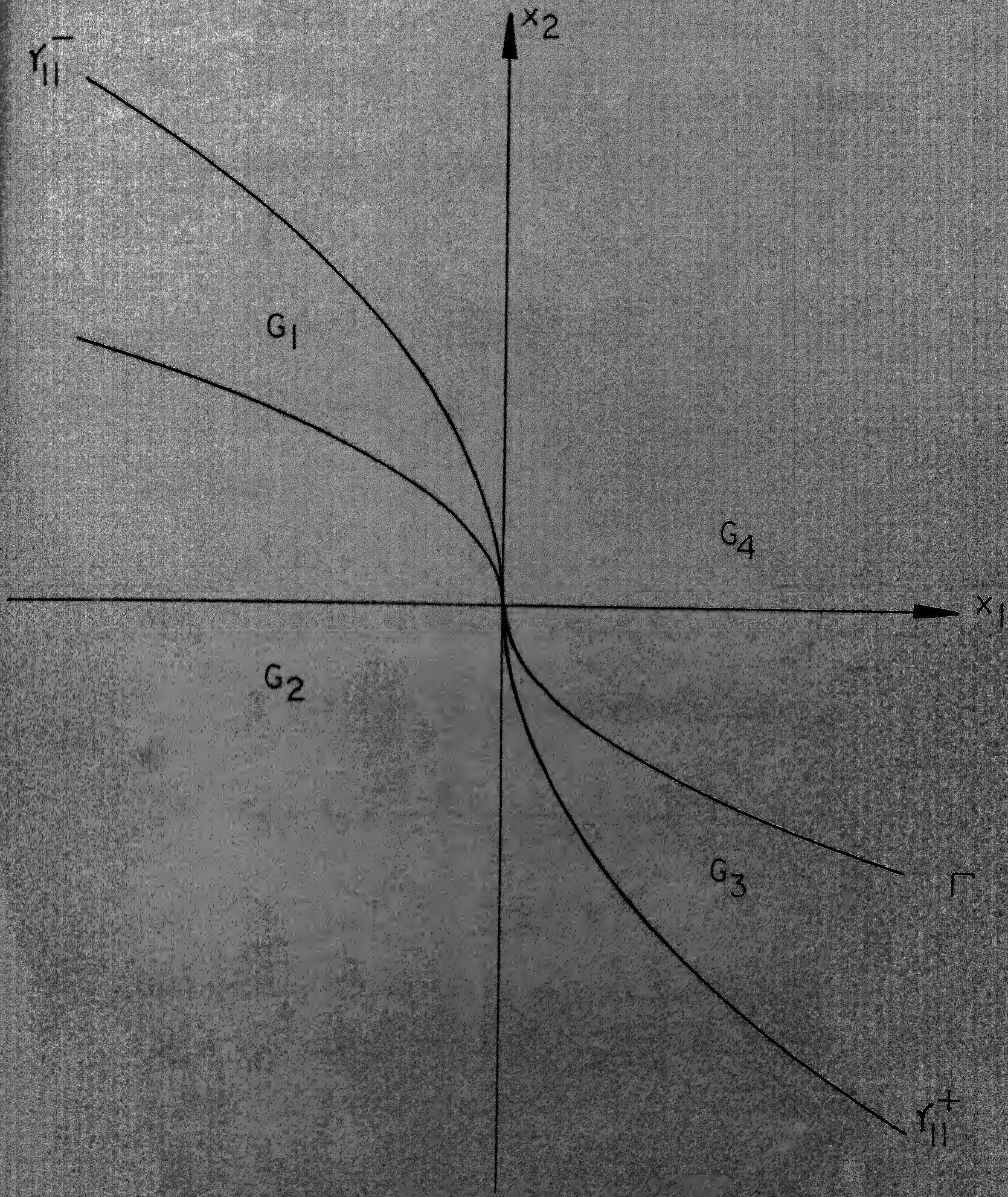


FIG.3.3 SWITCHING SURFACES FOR THE NONCOOPERATIVE SOLUTION

CASE $c > b$

The quantities in (3.85) refer to time t_3-. Solving (3.85) to be consistent with (3.71) and (3.66) yields

$$\lambda_1^1(t_3-) = -1/s_2$$

$$\lambda_2^1(t_3-) = 0$$

$$\lambda_1^2(t_3-) = -\frac{2+c+b}{(2+c)s_2} \tag{3.86}$$

$$\lambda_2^2(t_3-) = \frac{b}{2+c}$$

and that

$$u^1(t) = 1 \quad ; \quad u^2(t) = 0 \tag{3.87}$$

for t in the interval (t_2, t_3).

Now from (3.71) and (3.86), we have

$$t_3 - t_2 = \frac{\left(\frac{b}{2+c} + \frac{b}{c}\right)(2+c)s_2}{(2+b+c)}$$

$$= \frac{2b}{c} \cdot \frac{1+c}{2+b+c}\, s_2 \tag{3.88}$$

Integrating (3.62) with control law (3.87) for time $(t_3 - t_2)$ given by (3.88)

$$s_2 = r_2 + (t_3 - t_2)$$

$$s_1 = r_1 + r_2\,(t_3 - t_2) + \frac{1}{2}\,(t_3 - t_2)^2 \tag{3.89}$$

Solving (3.88), (3.89) and (3.83) together for the equation of the switching curve, we have

$$r_2 = \frac{(c-b)(2+c)}{c(2+b+c)} s_2$$

$$r_1 = -\frac{\alpha}{2} r_2^2 \qquad (3.90)$$

where

$$\alpha = \frac{(2+b+c)^2 c^2 + 4(1+c)^2(2+c)(c-b)b + 4(1+c)^3 b^2}{(1+c)(2+c)^2(c-b)^2} \qquad (3.91)$$

From (3.86) and (3.87) we have at this corner,

$$\lambda_1^1(t_2+) = -\frac{1}{s_2}$$

$$\lambda_2^1(t_2+) = \lambda_1^1(t_2+)(t_3 - t_2) \qquad (3.92)$$

$$= -\frac{2b(1+c)}{c(2+b+c)} \qquad (3.92)$$

$$\lambda_1^2(t_2+) = -\frac{2+b+c}{(2+c)s_2}$$

$$\lambda_2^2(t_2+) = -\frac{b}{c}$$

Applying the corner conditions (3.25) here, we get with $\sigma = (r_2, t_2)$

$$[\lambda_1^1(t_2-) + \frac{1}{s_2}](-\alpha r_2) + [\lambda_2^1(t_2-) + \frac{2b(1+c)}{c(2+b+c)}] = 0$$

$$1 + \lambda_1^1(t_2-)r_2 + \lambda_2^1(t_2-)(u^1 + cu^2) = 0$$

$$[\lambda_1^2(t_2-) + \frac{2+b+c}{(2+c)s_2}](-\alpha r_2) + [\lambda_2^2(t_2-) + \frac{b}{c}] = 0 \qquad (3.93)$$

$$|u^1| + b|u^2| + \lambda_1^2(t_2-)r_2 + \lambda_2^2(t_2-)(u^1 + cu^2) = 0$$

Solving (3.93), (3.71) and (3.66) consistently, we have

$$u^1(t) = u^2(t) = +1 \quad \text{for} \quad t < t_2 \tag{3.94}$$

for $t < t_2$.

Similarly the sequence $\begin{bmatrix} -1 & -1 & +1 \\ -1 & 0 & +1 \end{bmatrix}$ can be shown to be equilibrium optimal. The Noncooperative Optimal Control Law is summerized below according to the demarcation of the state-space in Figure 3.3, since the equilibrium sequences obtained above are unique for any starting point and the optimal paths cover the entire state space.

The switching surfaces γ_{11} and Γ are defined as

$$\gamma_{11} = \left\{ (x_1,x_2) : x_1 = - \frac{x_2^2 \operatorname{sgn} x_2}{2(1+c)} \right\} \tag{3.95}$$

$$\Gamma = \left\{ (x_1,x_2) : x_1 = - \frac{\alpha}{2} x_2^2 \operatorname{sgn} x_2 \right\} \tag{3.96}$$

where α is given by (3.91) and

$$\begin{array}{ll} (x_1,x_2) \in G_1 & (u^1,u^2) = (\ 1,\ 0) \\ (x_1,x_2) \in G_3 & (u^1,u^2) = (-1,\ 0) \\ (x_1,x_2) \in G_2 \cup \gamma_{11}^{+} & (u^1,u^2) = (\ 1,\ 1) \\ (x_1,x_2) \in G_4 \cup \gamma_{11}^{-} & (u^1,u^2) = (-1,-1) \end{array} \tag{3.97}$$

10

It should be noted that for the null information case, the optimal strategies are only functions of time

and the synthesis problem is solved for the perfect information case where the strategies are closed-loop control laws. The above solution can indeed be shown to be valid for the perfect information case by verifying the Hamilton-Jacobi equation (3.42).

The Noncooperative Value Function (W^1, W^2) are calculated using the general results (B.9) of Appendix B and tabulated in Table 3.1.

The Hamilton-Jacobi equation (3.42) is verified below for the Region G_1.

For player 1, we have

$$0 = \min_{|u^1|<1} \left\{ 1 - \sqrt{\frac{2+c}{1+c}} \cdot \frac{x_2}{\sqrt{x_2^2 - 2x_1}} - \left(1 - \sqrt{\frac{2+c}{1+c}} \cdot \frac{x_2}{\sqrt{x_2^2 - 2x_1}}\right)(u^1 + cu^2) \right\} \tag{3.98}$$

Now since

$$\begin{aligned} \sqrt{2+c}\; x_2 &= \sqrt{(1+c)x_2^2 + x_2^2} \\ &< \sqrt{(1+c)x_2^2 - 2x_1(1+c)} \\ &= \sqrt{1+c}\;\sqrt{x_2^2 - 2x_1} \end{aligned} \tag{3.99}$$

It follows that

$$u^1 = \mathrm{sgn}\left(1 - \sqrt{\frac{2+c}{1+c}}\; \frac{x_2}{\sqrt{x_2^2 - 2x_1}}\right) = +1 \tag{3.100}$$

TABLE 3.1

Value Function (W^1, W^2) of Noncooperative Solution

Region	W^1	W^2
1	$-x_2+\sqrt{\frac{2+c}{1+c}}\cdot\sqrt{x_2^2-2x_1}$	$-x_2+\frac{2+c+b}{\sqrt{(2+c)(1+c)}}\cdot\sqrt{x_2^2-2x_1}$
2	$-\frac{x_2}{1+c}+\frac{-c+\sqrt{(2+c)(1+c)(1+\alpha)}}{(1+c)\sqrt{1+\alpha(1+c)}}\times\sqrt{x_2^2-2x_1(1+c)}$	$-\frac{x_2(1+b)}{1+c}+\frac{b-c+(2+c+b)\sqrt{\frac{(1+\alpha)(1+c)}{2+c}}}{1+c}\times\sqrt{x_2^2-2x_1(1+c)}$
3	$x_2+\sqrt{\frac{2+c}{1+c}}\sqrt{x_2^2+2x_1}$	$x_2+\frac{2+c+b}{\sqrt{(2+c)(1+c)}}\sqrt{x_2^2+2x_1}$
4	$\frac{x_2}{1+c}+\frac{-c+\sqrt{(2+c)(1+c)(1+\alpha)}}{(1+c)\sqrt{1+\alpha(1+c)}}\times\sqrt{x_2^2+2x_1(1+c)}$	$\frac{x_2(1+b)}{1+c}+\frac{b-c+(2+c+b)\sqrt{\frac{(1+\alpha)(1+c)}{2+c}}}{1+c}\times\sqrt{x_2^2+2x_1(1+c)}$

Similarly for player 2, we have

$$0 = \min_{|u^2|<1} \left\{ |u^1|+b|u^2| - \frac{(2+c+b)x_2}{\sqrt{(2+c)(1+c)(x_2^2-2x_1)}} - \left(1- \frac{(2+c+b)x_2}{\sqrt{(2+c)(1+c)(x_2^2-2x_1)}}\right)(u^1+cu^2) \right. \tag{3.101}$$

Since

$$\frac{(2+c+b)x_2}{\sqrt{(2+c)(1+c)(x_2^2-2x_1)}} -1 < \frac{(2+c+b)\sqrt{(1+c)(x_2^2-2x_1)}}{(2+c)\sqrt{(1+c)(x_2^2-2x_1)}} -1$$

$$= \frac{b}{c+2} < \frac{b}{c} \tag{3.102}$$

it follows that

$$u^2 = \text{dez}\left(1- \frac{(2+c+b)x_2}{\sqrt{(2+c)(1+c)(x_2^2-2x_1)}}\right)\frac{c}{b} = 0 \tag{3.103}$$

Similarly it has been verified for the other regions. This example is solved in detail so that in future, we can relegate the unnecessary details to the appendices.

3.5 CONCLUSIONS

In this chapter, necessary and some sufficient conditions are given for a class of deterministic N-person differential games. The class of games is assumed not to exhibit singular and abnormal solutions. The application

of these conditions is shown through solving two examples. The second example has a much deeper significance and for the complete solution of this example, we should relax the above restriction which is the main theme of the next chapter.

The only types of information patterns assumed for the players are perfect and no observations. Though we continue with this assumption, we believe that observability concept plays an important role in the case of partial observations to the players, which means that their observation vectors are of dimension less than that of the state of the game:

We will have occasion to come across more than one equilibrium points in the next chapter. The noncooperative solution is to be defined in this case through the extra concepts given in Chapter II.

CHAPTER IV

SWITCHING SURFACES IN DIFFERENTIAL GAMES

4.1 INTRODUCTION

In the preceding chapter we considered a class of noncooperative differential games. In these the players were permitted to use discontinuous strategies and the discontinuities in their optimal strategies were assumed to lie on certain surfaces in the playing space $\mathcal{R}$ for the game. Further we imposed the condition that the game does not exhibit Singular and Abnormal solutions. The present chapter is devoted to a study of the surfaces containing such solutions. These, as well as the surfaces containing the discontinuities in the optimal strategies of the players, will be referred to as switching surfaces hereafter.

Since the strategy of any player with perfect information is a feedback control law, the solution of a game requires (loosely speaking) the compulsory solution of the synthesis problem for all the players with perfect information which is only optional in the case of optimal control problems. Perhaps one could also solve the synthesis problem for other players[1] for

1 The misleading term 'optimal open-loop feedback control' is used by some authors for this solution for players with no observations.

convenience in representing their control laws and the optimal paths in the playing space of the game. The construction of switching surfaces hold in either case with proper interpretation.

The solution of the game between the switching surfaces is obtained in a routine fashion by integrating the canonical equations obtained by the minimum principle stated in Chapter III. This is referred to as the 'solution in the small' (Isaacs 1965) in contrast to the 'solution in the large' consisting of the construction of the switching surfaces. This latter aspect is on an uneasy terrain in the literature and is mostly example-oriented (Breakwell 1969).

A general classification of the switching surfaces in optimal control and differential games is presented in Section 4.2. Next we deal with the conditions to be satisfied on them for their construction, along with simple examples. We present in Section 4.4 the complete noncooperative solution of the double-integral plant introduced in Chapter III.

4.2 CLASSIFICATION OF SWITCHING SURFACES

An exhaustive classification of switching surfaces can be made considering the nature of optimal paths on the surface and its immediate neighbourhood, i.e., whether

the optimal paths enter, leave or are parallel to the surface (not necessarily of the same nature on either side). Isaacs[2] resorted to this classification and showed that some of the surfaces under this classification are unrealizable. His Universal, Dispersal and Transition Surfaces derive their names on this basis. Any switching surface can be labelled as belonging to the players whose strategies are discontinuous across the surface.

A different classification is based on the method of construction or the condition to be satisfied on the switching surface. Thus Transition Surfaces are constructed by the application of corner conditions (3.24). The other candidates in this classification are the Singular, Dispersal and Abnormal Surfaces.

Singular Surfaces are surfaces containing singular optimal paths. A definition of Robbins (1967) is generalized here. An Extremal Arc is Singular, if at each point of the arc there is some allowable first-order weak control variation for at least one player p, which leaves his corresponding Hamiltonian H^p unchanged to second order. This condition reduces to $H^p_{u^p u^p}$ being singular, if

---

2 Isaacs (1965) uses the term Singular Surfaces in place of switching surfaces used in this thesis. We use the term Singular Surfaces to represent surfaces containing singular solutions.

the control vector u^p is interior to its restraint set Ω^p. In other words, on a singular arc the Legendre-Clebsch condition is satisfied in its weak form only. As in optimal control problems, the most common examples arise when the Hamiltonian of a player is linear or sectionally linear in some of his control variables. Most of the Universal Surfaces of Isaacs (1965) fall in this category. The occurance of these arcs in relaxed variational problems is well-known (Warga 1962).

The Dispersal Surface of a player consists of starting points in $\mathcal{R}$ corresponding to which the player has multiple equilibrium sequences yielding the same Value to him. These are thus identical with the $\mathcal{N}$ surfaces of the Regular Decomposition in Section 3.2. The Value Function of any other player whose strategy is continuous across this surface may be discontinuous because of the nonequivalence of the different equilibrium strategies. A thorough presentation of these surfaces for the two-person zero-sum case is given by Isaacs (1965).

Abnormal Surfaces are surfaces containing abnormal solutions. On the abnormal paths, the minimum principle in Chapter III is satisfied with $\lambda_o^p = 0$. Thus $\lambda_o^p = 0$ in the expression (3.20) for the Hamiltonian H^p and the Transversality and Corner Conditions (3.22) - (3.25). It

is well-known in optimal control that there may be no neighbouring curves with admissible controls in the case of abnormal solutions. The Abnormal Surfaces in problems with time as payoff have a special significance as is evidenced by the concept of Barrier in two-person zero-sum games studied by Isaacs (1965).

4.3 CONSTRUCTION OF THE SWITCHING SURFACES

The construction of switching surfaces in optimal control problems appears in contemporary literature. The familiar problems are the ones with the Hamiltonian linear or sectionally linear in the control variables which are bounded. The resulting bang-bang, three-level etc., controls are given in terms of signum, dead-zone etc., functions of a suitable Switching Function with the state and adjoint variables x, λ as its arguments. Under the usual smoothness assumptions on the formulation functions as given in Section 3.2 - f and L are assumed class $C^{(1)}$ - the Switching Function as well as λ are continuous functions of their respective arguments. In these problems the construction of switching surfaces is an easy matter.

On the other hand in Differential Games, the Switching Functions as well as the adjoint variables λ^p need not be continuous in spite of similar smoothness assumptions on f and L^p (see Example 3.2). This

situation arises because the discontinuities in the optimal strategies of the other players reflect in the dynamic equations of the one-sided optimal control problem of the remaining player p. The preceding result is true whatever be the information patterns to the players. The corner conditions stated by Berkovitz (1961) for such a problem are equivalent to the results in Section 3.2. In the same vein, it is shown in (Sarma et.al. 1969) that if all other players have continuous strategies at the switching surface of any player p, then his adjoint variables λ^p are continuous across this surface. It is for this reason that in Example 3.2, the adjoint variables for player 2 are continuous at the corner represented by time t_2 (see Figure 3.2).

The possible discontinuities in the adjoint variables together with the simultaneity involved in obtaining the strategies of all the players makes the construction of switching surfaces more difficult in differential games. This explains to a large extent the Bang-Bang-Bang surfaces, jocularly named so by Isaacs(1969). With these remarks, we indicate the construction of the specific surfaces below.

Singular Surfaces:

Singular extremals in optimal control have been studied in the literature (for example Johnson 1965,

Kelley et.al. 1966 and Robbins 1967) and their study is linked with the Hilbert Differentiability Condition of the classical variational calculus. Of these the Robbins' version of the second variation method is presented here so as to be applicable to linear singular arcs in Differential Games. Linear singular arcs arise because of the Hamiltonian H^p of a player being linear in some components of u^p and when u^p is interior to its restraint set. In the remaining cases, either the situation is too transparent or it can be dealt on similar lines as in optimal control problems.

The linear singular control variables cannot be determined from (3.37) of the minimum principle. However, differentiation of $H^p_{u^p}$ a sufficient number of times equal to the order of singularity will determine the singular control variables after suitable manipulation. The manipulation consists in substituting the canonical equations and the expressions for the nonsingular control variables after each differentiation. The generalized Legendre-Clebsch condition is stated as a test for the optimality of singular extremals.

Generalized Legendre-Clebsch Condition: For the player p, Φ^p_k is formed successively for different values of k, where

$$\Phi^p_k = [(\frac{d}{dt})^k \quad H^p_{u^p}]_{u^p} \tag{4.1}$$

The first time Φ_k^p is not equal to a null matrix, k should be even, i.e., $k = 2\ell$. The matrix $(-1)^{\ell}\,\Phi_{2\ell}^p$ must be positive semidefinite. If this condition fails, the singular extremal is not optimal. The two-person zero-sum version of this condition is given by Anderson (1969) along with a few junction conditions. These junction conditions are however applicable only when the remaining player uses continuous strategy across the junction.

If the different control variables have different orders of singularity, the test is applied successively and the variables are reduced in their order of singularity. If the control variables of several players are singular, the test should be applied simultaneously to all the corresponding Hamiltonians. Thus the actual construction of Singular Surfaces is accompanied with several analytical difficulties. We present here a simple example.

Example 4.1: We shall examine the possibility of singular solutions in Example 3.2. The game satisfies the equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u^1 + cu^2 \end{aligned} \tag{4.2}$$

referred to as the double integral plant with the two inputs constrained as follows.

$$|u^1| \leqslant 1 \quad ; \quad |u^2| \leqslant 1 \tag{4.3}$$

The payoff functionals of the players are given by

$$J^1[x_o, \underline{u}] = \int_0^{t_f} dt$$
$$J^2[x_o, \underline{u}] = \int_0^{t_f} \{|u^1| + b|u^2|\} dt \qquad (4.4)$$

where x_o is the initial state and the terminal state is specified as the origin. Equations (4.2) - (4.4) are identical with (3.62) - (3.64).

The application of the necessary conditions is shown in Section 3.4. The Hamiltonians, adjoint equations and the optimal control actions for this problem are given by (3.65), (3.71) and (3.66) respectively, and are reproduced below for ready reference.

$$H^1(x, \underline{u}, \lambda^1) = 1 + \lambda_1^1 x_2 + \lambda_2^1 (u^1 + cu^2)$$
$$H^2(x, \underline{u}, \lambda^2) = |u^1| + b|u^2| + \lambda_1^2 x_2 + \lambda_2^2(u^1 + cu^2) \qquad (4.5)$$

$$\dot{\lambda}_1^p = 0$$
$$\dot{\lambda}_2^p = -\lambda_1^p \qquad (4.6)$$

for $p = 1, 2$ and

$$u^{1*} = -\operatorname{sgn} \lambda_2^1$$
$$u^{2*} = -\operatorname{dez}(\lambda_2^2 \, c/b) \qquad (4.7)$$

The signum and deadzone functions in (4.7) are defined in (3.67) and (3.68) respectively.

A singular u^1 requires that $\lambda_2^1(t) = 0$ and $\dot{\lambda}_2^1(t) = -\lambda_1^1(t) = 0$ which violates the condition $H^1 = 0$ on the trajectory and hence does not arise. On the other hand, a singular u^2 requires

$$\frac{\lambda_2^2(t)\ c}{b} = \pm 1 \quad \text{or} \quad \lambda_2^2(t) = \pm\frac{b}{c} \tag{4.8}$$

and

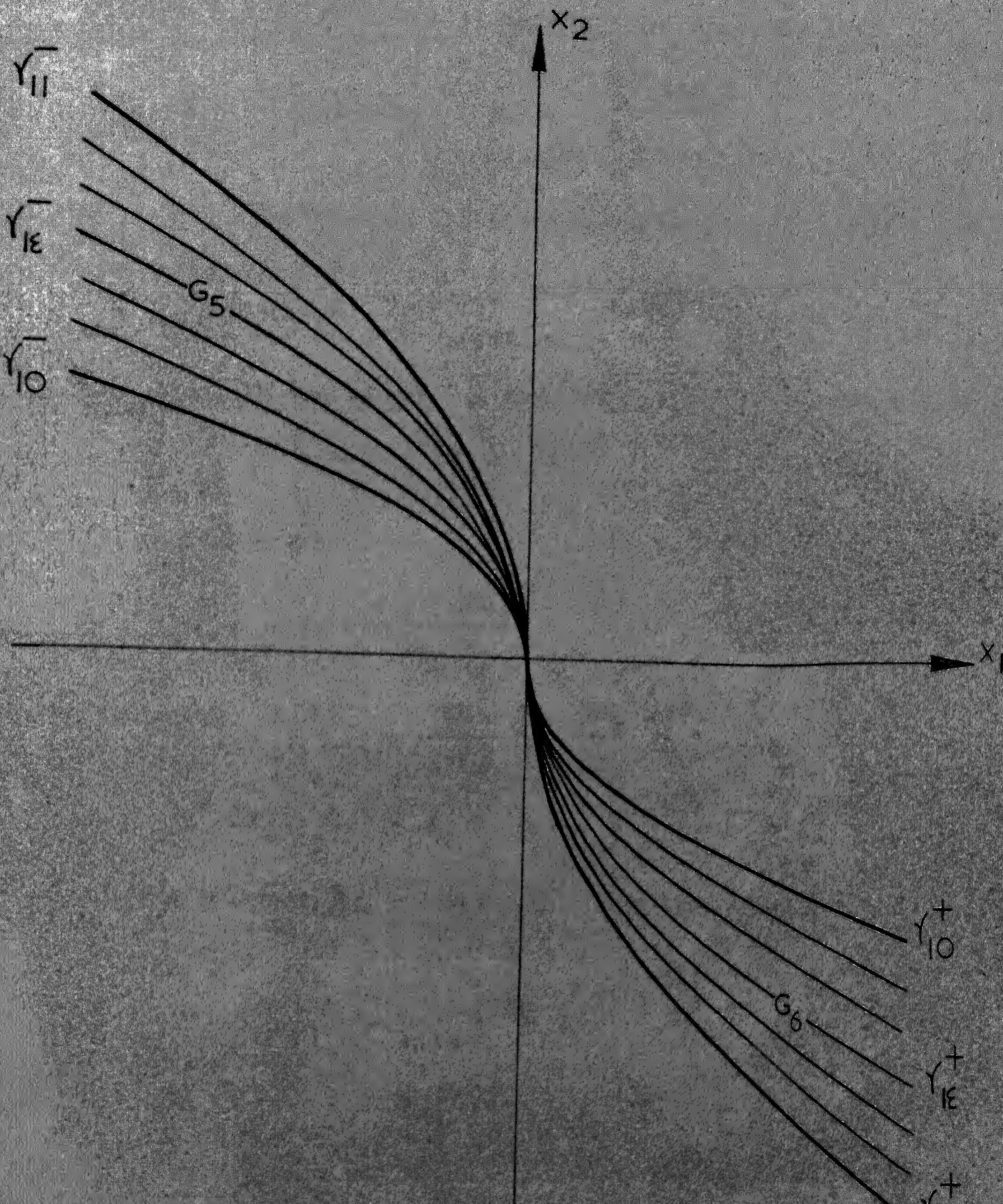
$$\dot{\lambda}_2^2(t) = -\lambda_1^2(t) = 0 \tag{4.9}$$

Thus for the terminal sequence $\begin{bmatrix} -1 \\ -\epsilon \end{bmatrix}$ with $0 \leqslant \epsilon \leqslant 1$ to be optimal, we should have from the transversality condition (3.76) and (4.8) (considering the upper values) and (4.9)

$$\lambda_1^1(t) = \lambda_1^2(t) = 0 \tag{4.10}$$

$$\lambda_2^2(t) = (1 + b\,\epsilon)\,\lambda_2^1(t) = -\frac{1 + b\,\epsilon}{1 + c\,\epsilon} = \frac{b}{c} \tag{4.11}$$

Equation (4.11) requires that $b = c$ and ϵ to be a constant on the trajectory. It can be seen that this sequence does not violate the Generalized Legendre-Clebsch Condition. Considering the lower values of (4.8) a similar result can be shown for the terminal sequence $\begin{bmatrix} +1 \\ +\epsilon \end{bmatrix}$. The resulting trajectories for $0 \leqslant \epsilon \leqslant 1$ are shown in regions G_5 and G_6 of Figure 4.1. The equation of the trajectory along which the state reaches origin with the control law $\begin{bmatrix} \pm 1 \\ \pm\epsilon \end{bmatrix}$



4.1 SINGULAR SOLUTIONS FOR THE EXAMPLE OF THE DOUBLE INTEGRAL PLANT.

with $0 \leq \epsilon \leq 1$ is given by (see Appendix A)

$$\gamma_{1\epsilon} = \left\{ (x_1, x_2) : x_1 = - \frac{x_2^2 \operatorname{sgn} x_2}{2(1 + c\epsilon)} \right\} \tag{4.12}$$

Now we can determine whether these trajectories are optimal for the perfect information case by verifying the Hamilton-Jacobi equation. Expressing the control laws as feedback policies in the Region G_5, we have in view of (4.12) and Appendix A

$$\begin{aligned} U^1 &= -1 \\ U^2 &= -\epsilon = \frac{x_2^2 + 2x_1}{2x_1 c} \end{aligned} \tag{4.13}$$

$$\begin{aligned} W^1 (x_1, x_2) &= \frac{x_2}{1+c\epsilon} = - \frac{2x_1}{x_2} \\ W^2 (x_1, x_2) &= x_2 \end{aligned} \tag{4.14}$$

Now for player 1, we have the Hamilton-Jacobi equation

$$0 = \min_{|u^1| < 1} \left\{ 1 + \left(- \frac{2}{x_2}\right) x_2 + \frac{2x_1}{x_2^2} (u^1 + cU^2) \right\}$$

or

$$u^{1*} = - \operatorname{sgn} \frac{2x_1}{x_2^2} = + 1 \tag{4.15}$$

Since (4.15) contradicts (4.13), the cluster of singular solutions in G_5 (also G_6 similarly) are not optimal for Player 1.

However by writing the Hamilton-Jacobi equation for Player 2, it can be easily seen that the cluster of

singular solutions in G_5 and G_6 are optimal for this player. This result finds application in the next section.

Dispersal Surfaces :

In optimal control problems, with the usual smoothness assumptions on f and L and the terminal surface, Dispersal Surfaces are not usually met with. Since for any player, having multiple strategies at a Dispersal Surface, the Values corresponding to all these are equal, these surfaces are constructed based on this property (Isaacs 1965). However on the Dispersal Surface $\mathcal{N}_{i_1, \ldots i_k}$ of player p the following condition holds. For any dx, dt variations on the manifold (Berkovitz 1964), we have

$$\begin{aligned} H^p(x, \lambda^p_{i_1}, \underline{U}^*_{i_1}, t) - \lambda^p_{i_1}\, dx &= H^p(x, \lambda^p_{i_2}, \underline{U}^*_{i_2}, t) - \lambda^p_{i_2}\, dx \\ &\;\vdots \qquad\qquad\qquad\qquad\qquad\qquad (4.16) \\ &= H^p(x, \lambda^p_{i_k}, \underline{U}^*_{i_k}, t) - \lambda^p_{i_k}\, dx \end{aligned}$$

Example 4.2 : We construct the Dispersal Surface for the double integral plant (4.2) and (4.3) on a restricted playing space. The playing space $\mathcal{R}$ and the terminal surface $\mathcal{T}_1 \cup \mathcal{T}_2$ are shown in Figure 4.2. $\mathcal{T}_1$ and $\mathcal{T}_2$ are given by

$$\begin{aligned} \mathcal{T}_1 &= \left\{(x_1, x_2) : x_1 = -\frac{x_2^2}{2(1+c)}\right\} \\ \mathcal{T}_2 &= \left\{(x_1, x_2) : x_1 = -\frac{x_2^2}{2}\right\} \end{aligned} \qquad (4.17)$$

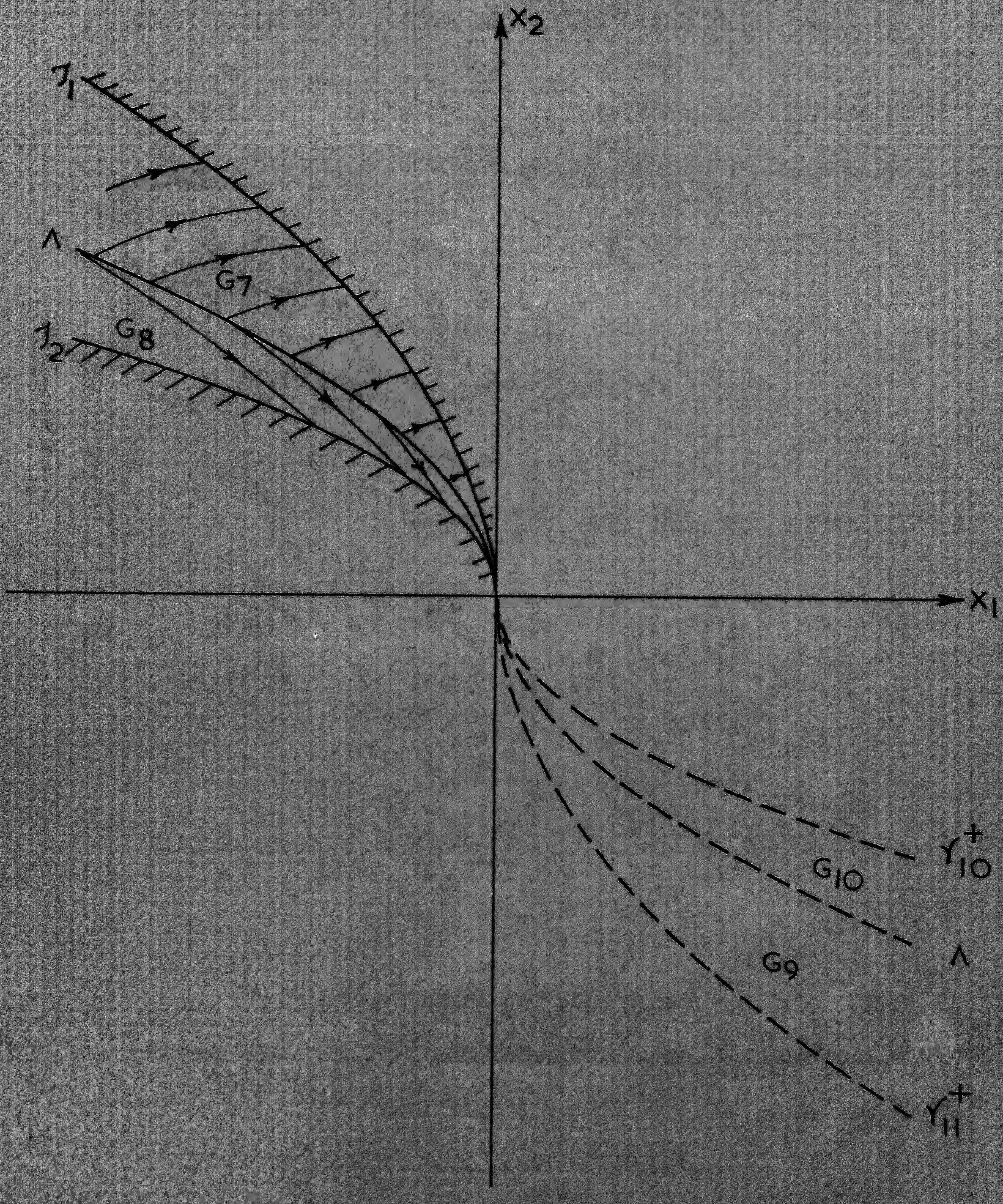


FIG. 4.2 AN EXAMPLE OF A GAME WITH A DISPERSAL SURFACE.

Thus $\mathcal{J}_1$ and $\mathcal{J}_2$ are identical with γ_{11}^- and γ_{10}^- respectively (see Appendix A).

The payoff functionals are of the composite type given by

$$J^1[x_0, \underline{u}] = \phi^1(x_f) + \int_0^{t_f} dt$$
$$J^2[x_0, \underline{u}] = \phi^2(x_f) + \int_0^{t_f} \left\{ |u^1| + b|u^2| \right\} dt \qquad (4.18)$$

where x_f is the terminal state at the free terminal time t_f. The functions ϕ^1 and ϕ^2 are given below.

$$\phi^1(x_f) = \begin{cases} \frac{x_{2f}}{1+c} & x_f \in \mathcal{J}_1 \\ x_{2f} & x_f \in \mathcal{J}_2 \end{cases} \qquad (4.19)$$

$$\phi^2(x_f) = \begin{cases} x_{2f}\left(\frac{1+b}{1+c}\right) & x_f \in \mathcal{J}_1 \\ x_{2f} & x_f \in \mathcal{J}_2 \end{cases} \qquad (4.20)$$

The Hamiltonians for the players, the adjoint equations and the optimal control actions are given again as in (4.5), (4.6) and (4.7) respectively. By the application of transversality conditions on $\mathcal{J}_1$, the optimal control sequence for paths that end on $\mathcal{J}_1$ is given by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as shown in (3.84) - (3.87). By a similar

application on $\mathcal{J}_2$, we have

$$
\begin{aligned}
&1 - \lambda_1^1(-x_{2f}) - \lambda_2^1 = 0 \\
&1 + \lambda_1^1 x_{2f} + \lambda_2^1 (u^1 + c\,u^2) = 0 \\
&1 - \lambda_1^2 (-x_{2f}) - \lambda_2^2 = 0 \\
&|u^1| + b|u^2| + \lambda_1^2 x_{2f} + \lambda_2^2(u^1 + c\,u^2) = 0
\end{aligned}
\tag{4.21}
$$

where all the quantities correspond to time t_f. Solving (4.21), (4.6) and (4.7) consistently under the imposed condition $c > 2$, we have

$$
\begin{aligned}
&\lambda_1^1(t_f) = -\frac{1}{x_{2f}} \quad ; \quad \lambda_2^1(t_f) = 0 \\
&\lambda_1^2(t_f) = \frac{-1 - b - \frac{b(1-c)}{c-2}}{x_2} \\
&\lambda_2^2(t_f) = \frac{b}{c-2}
\end{aligned}
\tag{4.22}
$$

$$
u^1(t) = 1 \quad ; \quad u^2(t) = -1 \quad \text{for} \quad t < t_f \tag{4.23}
$$

Thus when $c > 2$, the optimal sequence reaching $\mathcal{J}_2$ is $\begin{bmatrix} +1 \\ -1 \end{bmatrix}$ and when $c \leqslant 2$ there are no paths reaching $\mathcal{J}_2$.

Thus when $c > 2$, while the strategy of player 1 is continuous in $\mathcal{R}$, the second player's strategy is discontinuous and hence has a switching surface which in this case is a Dispersal Surface. For starting points on this surface W^2 must be same whether the optimal paths

reach $\mathcal{J}_1$ or $\mathcal{J}_2$. Thus making use of (B.9) of Appendix B, we have for any (x_1, x_2) on the Dispersal Surface

$$-\frac{x_2(1+b)}{1-c} + \sqrt{\frac{x_2^2-2x_1(1-c)}{1+(1-c)}}\left[1-\frac{1+b}{c-1}\right] = -x_2 + \frac{2+c+b}{\sqrt{(1+c)(2+c)}}\sqrt{x_2^2-2x_1} \tag{4.24}$$

Thus the Dispersal Surface is given by

$$\Lambda = \left\{(x_1, x_2) : x_1 = -\frac{\eta}{2}\, x_2^2\right\} \tag{4.25}$$

where η is given by

$$\frac{b+1}{c-1} - \frac{2+b-c}{c-1}\sqrt{\frac{-1+(c-1)\eta}{c-2}} = -1 + \frac{2+c+b}{\sqrt{(1+c)(2+c)}}\sqrt{1+\eta} \tag{4.26}$$

One can easily see that W^1 is not the same for both the optimal paths. Hence the two equilibrium points are nonequivalent for Player 1. There is no counterpart of this result in two-person zero-sum games for obvious reasons.

A similar construction can be made in the space between γ_{10}^{+} and γ_{11}^{+} as shown by the broken lines in Figure 4.2.

Abnormal Surfaces :

Abnormal solutions have not been extensively studied even in optimal control problems. Often the necessary and sufficient conditions for normality in the calculus of variations are difficult to translate into

optimal control theory. They are usually constructed from their definition. The Semipermeable Surfaces of Isaacs (1965) follow this construction and are examples of Abnormal Surfaces. The optimality of these solutions can be established easily only for the optimal control problems. In the case of two-person zero-sum games with time as payoff, these surfaces have a special significance associated with them, viz., the Barrier Concept and the roles of the players in determining these surfaces are fixed. We shall show, in terms of an example, the construction of Abnormal Solutions.

Example 4.3 : We construct the Abnormal solutions for the double integral plant problem given in (4.2) - (4.4). Since $\lambda_o^1 = \lambda_o^2 = 0$ in this case, (4.5) - (4.7) get modified as follows :

$$H^1 = \lambda_1^1 x_2 + \lambda_2^1 (u^1 + c u^2)$$
$$H^2 = \lambda_1^2 x_2 + \lambda_2^2 (u^1 + c u^2) \qquad (4.27)$$

$$\dot{\lambda}_1^p = 0$$
$$\dot{\lambda}_2^p = - \lambda_1^p \qquad (4.28)$$

for $p = 1,2$ and

$$u^{1*} = - \operatorname{sgn} \lambda_2^1$$
$$u^{2*} = - \operatorname{sgn} \lambda_2^2 \qquad (4.29)$$

The curves γ_{11}^{+}, γ_{11}^{-}, $\gamma_{1,-1}$, $\gamma_{-1,1}$ are the abnormal curves with the corresponding control sequences as $\begin{bmatrix}+1\\+1\end{bmatrix}$, $\begin{bmatrix}-1\\-1\end{bmatrix}$, $\begin{bmatrix}+1\\-1\end{bmatrix}$ and $\begin{bmatrix}-1\\+1\end{bmatrix}$. We show this below for γ_{11}^{-} by showing that it satisfies the required necessary conditions. The others follow similarly.

From the transversality conditions (3.24)[3] we have λ_1^1 and λ_1^2 arbitrary and

$$\lambda_2^1 (t_f) = \lambda_2^2 (t_f) = 0 \tag{4.30}$$

By (A.6) of Appendix A, we have for any initial state (x_1, x_2) on γ_{11}^{-}

$$t_f = \frac{x_2}{1+c} \tag{4.31}$$

From (4.28), (4.30) and (4.31) on integration, we have

$$\lambda_2^p (0) = \lambda_1^p \frac{x_2}{1+c} \tag{4.32}$$

Equations (4.32) and (4.29) yield

$$u^1 = u^2 = -1 \tag{4.33}$$

with λ_1^p assumed any negative number for $p = 1, 2$.

The solutions γ_{11}^{+} and γ_{11}^{-} will be shown to be optimal in the next section under certain conditions. This

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3 Equations (3.73) and (3.74) get modified with null vectors on the right hand side in this case.

is because both the players are primarily interested in terminating the game. The remaining curves $\gamma_{1,-1}$ and $\gamma_{-1,1}$ are probably optimal if we change the role of either player to maximize his payoff functional instead of the minimization that is assumed.

There are other switching surfaces in the literature which are constructed with the ideas presented here. For example, the Equivocal Surface (Isaacs 1965) is an Abnormal Surface which is also a Dispersal Surface corresponding to one player and a Singular Surface corresponding to the other. It is a member of a class of parametrized Abnormal Surfaces and is determined by the conditions corresponding to its being a Dispersal and a Singular Surface. As we stated earlier, the construction of these switching surfaces is mainly example-oriented. In the next section we make use of all the examples in this section to obtain the complete solution of the double integral plant problem.

4.4 NONCOOPERATIVE SOLUTION OF THE DOUBLE INTEGRAL PLANT

Here we present the complete solution of the problem formulated as Example 3.2. Its solution for the case $c > b$ is already presented there. For this case, there is a unique Nash equilibrium sequence for every starting point and this is defined as the noncooperative solution. We

showed that the solution is essentially the same for the cases when the players have no observations as well as when they have perfect knowledge of the state of the game.

The remaining cases $c \lessgtr b$ are solved in this section. We presently see that there is nonuniqueness of equilibrium sequences for certain starting points and the noncooperative solution is to be defined by a suitable selection of these sequences.

Case (1) $c < b$ and $2c - b^2 + c^2 + 2bc > 0$:

A procedure similar to that in Example 3.2 yields (see Appendix C) that the sequences $\begin{bmatrix} \pm 1 & \pm 1 & \mp 1 \\ \pm 1 & 0 & 0 \end{bmatrix}$ are in Nash equilibrium under the condition

$$\alpha' < \frac{1}{1+c} \tag{4.34}$$

where

$$\alpha' = \frac{b^2 - c^2 - 2bc}{(b - c)^2} \tag{4.35}$$

Condition (4.34) is equivalent to the assumption $2c - b^2 + c^2 + 2bc > 0$. This control law can be stated as follows in accordance with Figure 4.3(ii).

$$\gamma_{10} = \left\{(x_1, x_2) : x_1 = - \frac{x_2^2}{2} \operatorname{sgn} x_2\right\} \tag{4.36}$$

$$\Gamma' = \left\{ (x_1, x_2) : x_1 = - \frac{\alpha' x_2^2}{2} \operatorname{sgn} x_2\right\} \tag{4.37}$$

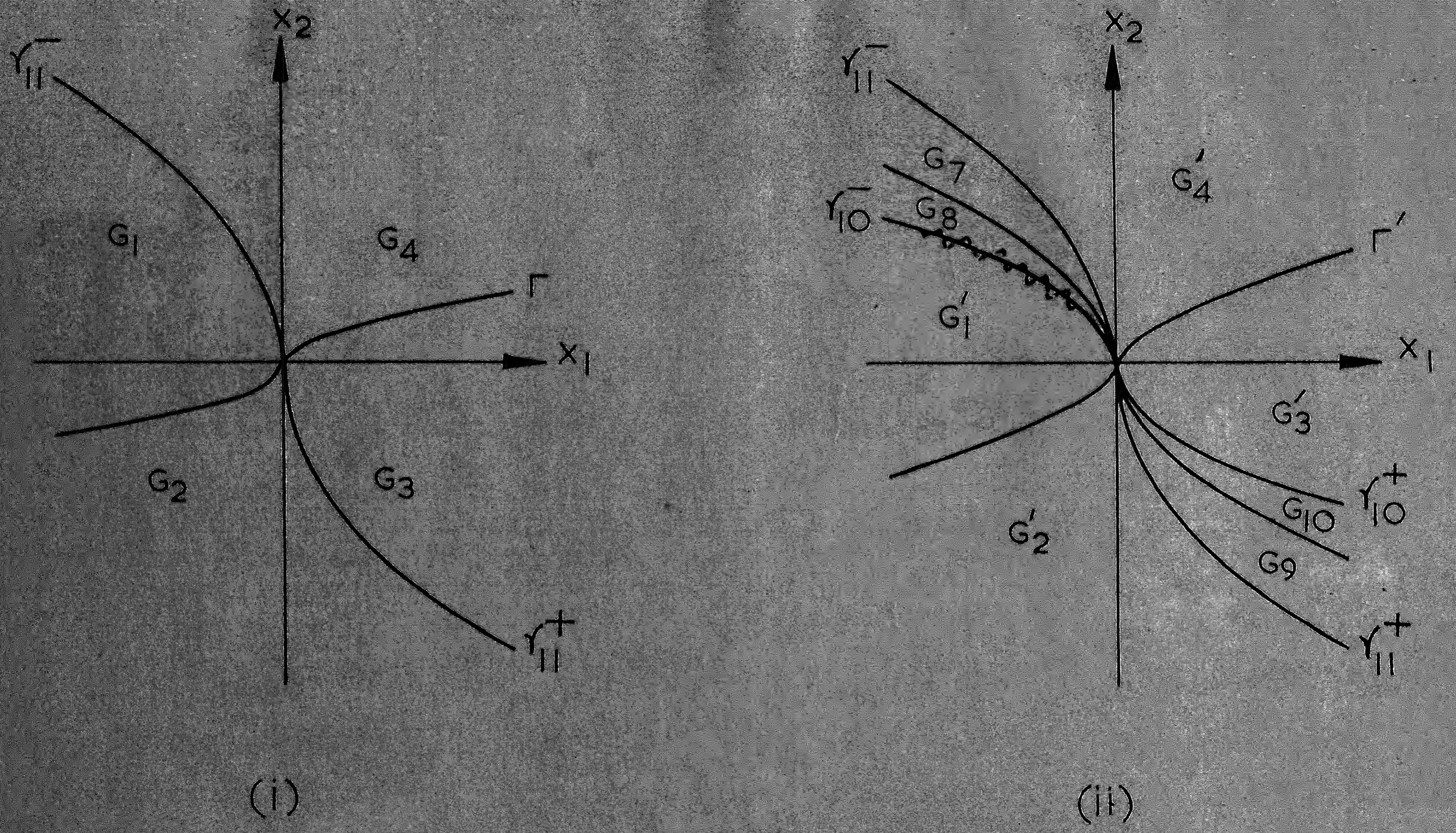

FIG.4.3 SWITCHING CURVES FOR NONCOOPERATIVE SOLUTION (PREFERENCES OF PLAYERS 1 AND 2)
CASE $c < b$ AND $2c - b^2 + c^2 + 2bc > 0$

$$(x_1,x_2) \in G_1 \qquad (u^1,u^2) = (1, 0)$$
$$(x_1,x_2) \in G_3 \qquad (u^1,u^2) = (-1, 0)$$
$$(x_1,x_2) \in G_2 \cup \Upsilon^+_{11} \qquad (u^1,u^2) = (1, 1) \qquad (4.41)$$
$$(x_1,x_2) \in G_4 \cup \Upsilon^-_{11} \qquad (u^1,u^2) = (-1,-1)$$

Considering Υ_{10} as the terminal surface, we can construct the equilibrium sequences $\begin{bmatrix} \pm 1 & \mp 1 \\ \mp 1 & 0 \end{bmatrix}$ under the assumption $c > 2$. No such sequences however exist when $c \leq 2$. This was shown in Example 4.2 where Υ_{10} can be identified with $\mathcal{J}_2$. In the same section we also constructed a Dispersal Surface Λ for the second player when $c > 2$. All this is shown in Figure 4.3(ii). Thus we have

$$(x_1,x_2) \in G_7 \qquad (u^1,u^2) = (1, 0)$$
$$(x_1,x_2) \in G_9 \qquad (u^1,u^2) = (-1, 0)$$
$$(x_1,x_2) \in G_8 \qquad (u^1,u^2) = (1,-1) \qquad (4.42)$$
$$(x_1,x_2) \in G_{10} \qquad (u^1,u^2) = (-1, 1)$$

Now we have to define the noncooperative solution taking into account the various equilibrium sequences $\begin{bmatrix} \pm 1 & \pm 1 & \mp 1 \\ \pm 1 & 0 & \mp 1 \end{bmatrix}$, $\begin{bmatrix} \pm 1 & \pm 1 & \mp 1 \\ \pm 1 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} \pm 1 & \mp 1 \\ \mp 1 & 0 \end{bmatrix}$ and the decomposition associated with them as shown in Figure 4.3. We can easily see that control sequences corresponding to

Figure 4.3(i) are preferred by Player 1 and those corresponding to Figure 4.3(ii) are preferred by Player 2. To illustrate, for any starting point (x_1, x_2) on Υ_{10}^{-}, we can compare the Values to both the players corresponding to both the cases of Figure 4.3. The Values can be calculated from (A.6) and (B.9) of Appendices A and B. Thus for Player 1 we have to verify that

$$- x_2 + \sqrt{\frac{x_2^2 - 2x_1}{1 + \frac{1}{1+c}}} \left[\frac{1}{1+c} + 1 \right] < x_2 \qquad (4.43)$$

or that

$$\sqrt{\frac{2(2+c)}{1+c}} - 1 < 1 \quad \text{i.e.} \quad 4+2c < 4+4c \qquad (4.44)$$

which is true since $c > 0$. Similarly for Player 2, we have to verify that

$$- x_2 + \sqrt{2}\, x_2 \sqrt{\frac{1+c}{2+c}} \, [1 + 1] > x_2 \qquad (4.45)$$

or that $\frac{1+c}{2+c} > \frac{1}{2}$ which is true since $c > 0$. The stated preferences of the players follow similarly.

No Observations to the Players : We can see from Figure 4.3 that the players have full agreement over the Regions G_7 and G_9 and this constitutes the noncooperative solution on these regions. As remarked earlier these regions extend completely between Υ_{10} and Υ_{11} when $c \leqslant 2$.

For other regions, the strategies are not interchangeable and hence present a coordination problem. The

game cannot be reduced using the Payoff Dominance Concept, since of the two equilibrium strategies which are noninterchangeable, one is preferred by one player and the other by the remaining player. Risk Dominance also does not apply since the recombined strategies are not playable[4]. This Deadlock cannot thus be resolved as in finite games. It may be possible to use mixed strategies to define the Tacit and Vocal solutions in this case. This problem is suggested for future research.

Perfect Information to the Players : Since the players have a running knowledge of the state, it is helpful to them in terminating the game in which both are interested. The problem cited above does not arise here since the recombined strategies are playable. Thus we can calculate the Risk to the players when each of them sticks to their preferred strategies, i.e., player 1 plays the strategy indicated by Figure 4.3(i) and player 2 that indicated by Figure 4.3(ii).

As an example, for any starting point (x_1, x_2) on γ_{10}, it is easy to see that the state of the game follows a chattering path to the origin along γ_{10}^- (shown exaggerated by broken lines in Figure 4.3(ii). The total time taken for

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4 This shows that they are derived from two different Normal Forms of the game.

this is x_2, same as the cost to Player 1 corresponding to the equilibrium strategies preferred by player 2, shown in Figure 4.3(ii).

Of the total time, for the fraction $\frac{c-2}{c}$ the players use $\begin{bmatrix} +1 \\ 0 \end{bmatrix}$ and for the remaining time they play $\begin{bmatrix} +1 \\ -1 \end{bmatrix}$ since

$$\frac{c-2}{c} \cdot 1 + \frac{2}{c}(1-c) = -1 \tag{4.46}$$

The cost to Player 2 corresponding to this chattering path is

$$x_2 \left[\frac{c-2}{c} \cdot 1 + \frac{2}{c}(1+b) \right] = (1+ \frac{2b}{c})\, x_2 \tag{4.47}$$

We also calculate from (B.9) of Appendix B the cost to Player 2 corresponding to the equilibrium strategies represented by Figure 4.3(i). It is given by

$$-x_2 + \frac{2+c+b}{\sqrt{(2+c)(1+c)}} \sqrt{x_2^2 + 2x_1}$$

$$= x_2 \frac{(2+c+b)\sqrt{2}}{\sqrt{(2+c)(1+c)}} - 1 \tag{4.48}$$

On simplification, (4.48) is less than (4.47). Hence while there is no risk involved for Player 1, there is considerable risk for Player 2 in adhering to his preferred strategy if the opponent also does the same. He derives a value inferior even to the equilibrium point preferred by his opponent.

Hence the noncooperative solution for this case is as represented by Figure 4.3(i). This solution consists of both normal and abnormal arcs.

Case (2) $c < b$ and $2c - b^2 + c^2 + 2bc < 0$:

In this case the equilibrium sequences are $\begin{bmatrix} \pm 1 & \mp 1 \\ 0 & 0 \end{bmatrix}$ as shown in Appendix C and the switching curve Γ' goes into the space between γ_{11} and γ_{10} (see Figure 4.4). The abnormal solution γ_{11} no longer satisfies the Envelope Principle of Isaacs (1965) and hence does not correspond to Nash equilibrium. In the region between Γ' and γ_{10}, there are no equilibrium solutions reaching Γ'. However for the case $c > 2$, we have $\begin{bmatrix} \pm 1 \\ \mp 1 \end{bmatrix}$ as equilibrium sequences corresponding to solutions reaching γ_{10} as shown in Example 4.2.

Perfect Observations to the Players : An interesting strategy available to player 2 to reach Γ' when the game is in between γ_{10} and Γ' is to use $u^2 = 0$. Since player 2 thus leaves the optimization problem entirely to his opponent, we call this his Abstaining Strategy. Player 1 through his observations can detect this and the best strategy available to him under this condition is to play $u^1 = \pm 1$ depending upon whether the state of the game is below or above γ_{10}. Now when $c > 2$ one can fit a Dispersal Surface Λ' for Player 2 separating his

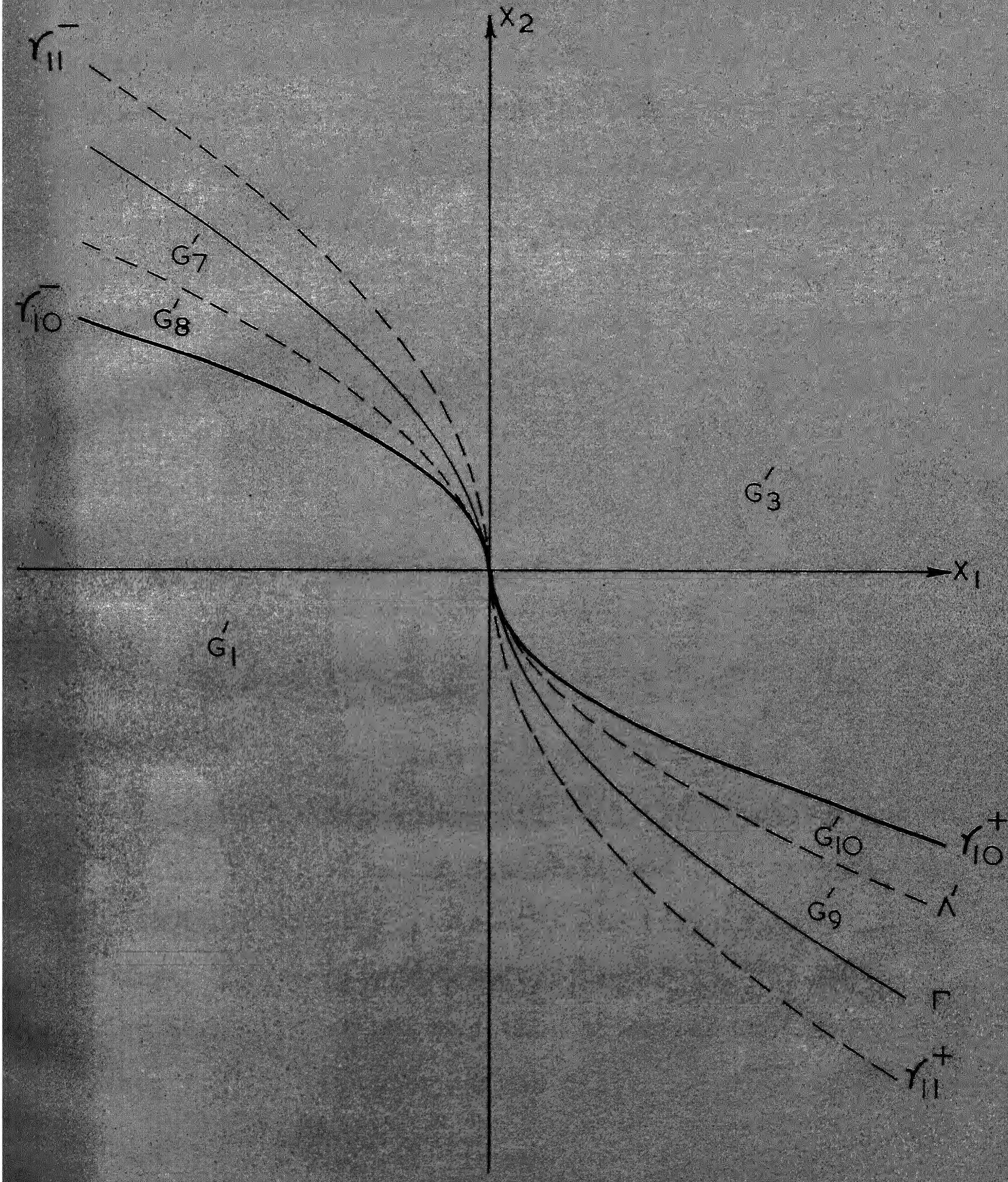


FIG. 4.4 SWITCHING SURFACES FOR THE NONCOOPERATIVE SOLUTION CASE $c > b$ AND $2c - b^2 + 2bc + c^2 < 0$.

Abstaining Strategy and the strategy $\begin{bmatrix} \pm 1 \\ \mp 1 \end{bmatrix}$ leading to γ_{10}. However this strategy is risky to Player 2 in the same way as discussed in Case (1). Thus the solution consists of the Player 2 completely abstaining from the play and leaving optimization to Player 1. Thus in the resulting time optimal problem, Player 1 uses $u^1 = +1$ in the region below γ_{10} and $u^1 = -1$ above γ_{10} in Figure 4.4. It may be noted that except in the regions between γ_{10} and Γ' where the game has no equilibrium point, the rest of the strategies are in equilibrium.

No Observations to the Players : Since the game does not exhibit multiplicity of equilibrium sequences, the above solution is valid except in the Regions between γ_{10} and Γ'.

Case (3) $c < b$ and $2c - b^2 + c^2 + 2bc = 0$:

This separates Cases (1) and (2). Considering in the limit the solutions of Cases (1) and (2) for this case, it is obvious that the Abstaining Strategy is suited best for Player 2 in the perfect information case. As in the earlier cases, it is not possible to define completely the solution for the no observations case.

Case (4) $c = b$:

In Example 4.1, we observed that the sequences $\begin{bmatrix} \pm 1 \\ \pm \epsilon \end{bmatrix}$ with $0 \leqslant \epsilon \leqslant 1$ qualify as equilibrium terminal sequences as they do not violate the Generalized Legendre-Clebsch

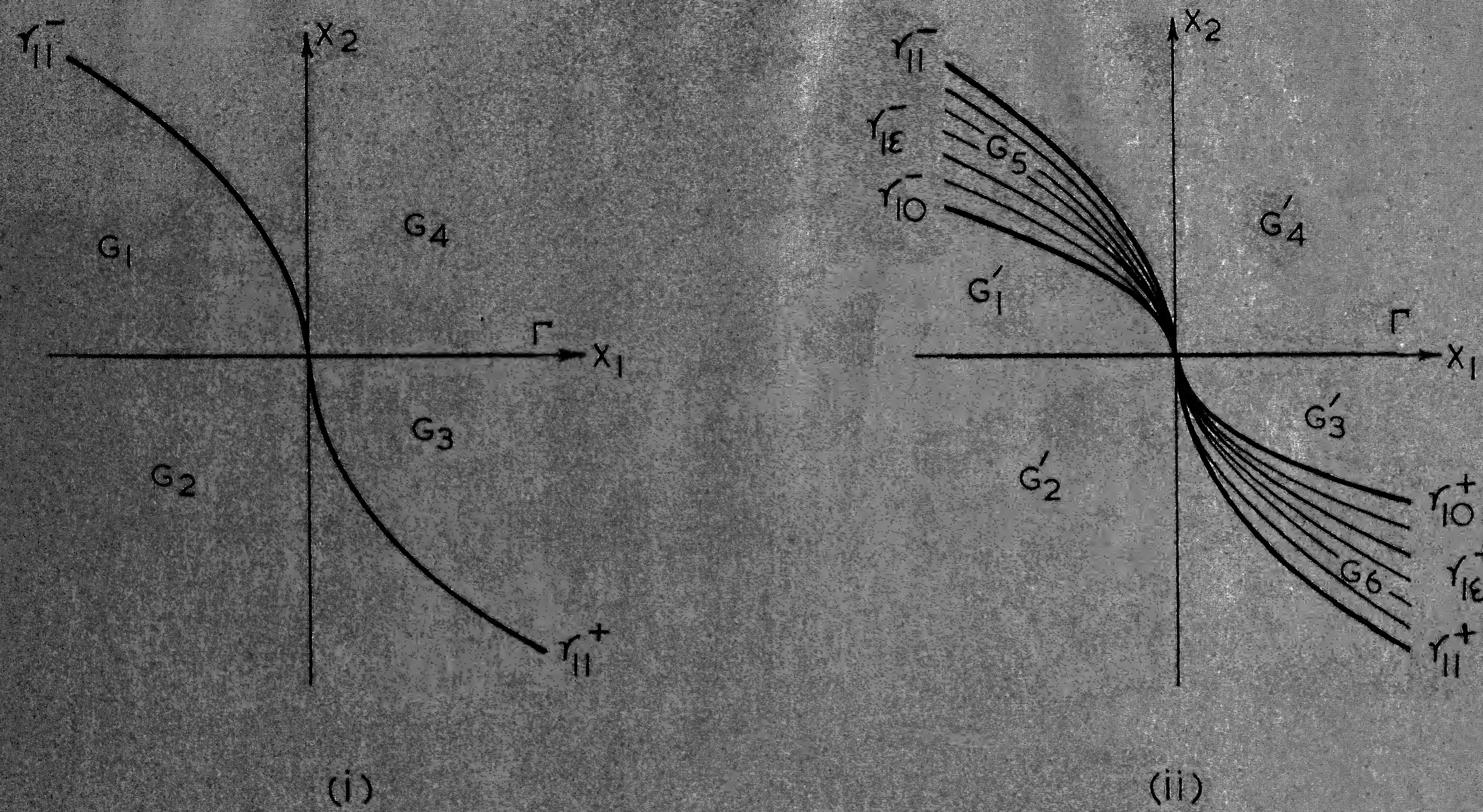
Condition. Thus one encounters an infinite number of equilibrium sequences $\begin{bmatrix} \pm 1 & \pm 1 & \mp 1 \\ \pm 1 & 0 & \mp \epsilon \end{bmatrix}$ with $0 \leqslant \epsilon \leqslant 1$ and $\begin{bmatrix} \pm 1 & \mp 1 \\ \mp 1 & \mp \epsilon \end{bmatrix}$ for $\epsilon < \frac{c-2}{c}$ (see Appendix C).

The Payoff Dominance concept applied to these equilibrium sequences yields that Player 1 prefers the sequences $\begin{bmatrix} \pm 1 & \pm 1 & \mp 1 \\ \pm 1 & 0 & \mp 1 \end{bmatrix}$ with the corresponding decomposition shown in Figure 4.5(i). Similarly for Player 2, the preferred decomposition is represented in Figure 4.5(ii). The switching curves Γ and Γ' given by (4.39) and (4.37) to this limiting case become identical with the x_1 - axis.

No Observations to Players : For starting points on γ_{11} there is full agreement between the players. Since the recombination of the individually preferred strategies are not playable for any other starting points, the game ends in a deadlock as pointed out in the earlier cases.

Perfect Observations to the Players : As in Case (1) the recombined strategies are playable in this case. Also similarly it is risky to play this strategy for Player 2 in the same sense. A repetition of the arguments in Case (1) gives the solution in the present case as given by Figure 4.5(i).

We give below a brief summary of the example and its solution as presented in Example 3.2 and this section.



(i) (ii)

FIG. 4.5 SWITCHING CURVES FOR NONCOOPERATIVE SOLUTION (PREFERENCES OF PLAYERS 1 AND 2) CASE $C = b$

Summary of the Results :

The game satisfies the equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u^1 + cu^2 \end{aligned} \tag{4.49}$$

The state of the game is to be transferred from x_0 to the origin by the control actions u^1, u^2 of the players which are constrained as follows :

$$|u^1| \leq 1 \quad ; \quad |u^2| \leq 1 \tag{4.50}$$

The payoff functions of the players are given by

$$\begin{aligned} J^1[x_0, \underline{u}] &= \int_0^{t_f} dt \\ J^2[x_0, \underline{u}] &= \int_0^{t_f} \{ |u^1| + b|u^2| \} \, dt \end{aligned} \tag{4.51}$$

which the respective players wish to minimize.

The solution of the game is defined below for the perfect information case. The solution of the game with no observations is defined only for the case $c > b$ and is essentially the same (except for implementation) as the corresponding case with perfect observations.

The equilibrium sequences for any starting point are unique when $c > b$ and the solution and the switching surfaces are indicated in Figure 3.3. The equilibrium sequences are nonunique for the case $c < b$ with the added

condition $2c - b^2 + c^2 + 2bc > 0$ and the case $c = b$. In particular there are uncountable number of sequences for the latter case. In these cases the solution is defined by a suitable selection of the sequences based on the concepts of Payoff and Risk Dominance. The switching curves for these solutions are represented in Figures 4.3(i) and 4.5(i).

The resulting solution in the above cases is essentially similar with the switching curves given by similar expressions. The switching curve lies in II and IV quadrants of the state space when $c > b$ and gradually becomes closer and coincides with the x_1 - axis when $c = b$. With $c < b$ it changes quadrants.

When $c < b$ and $2c - b^2 + c^2 + 2bc < 0$, there are no equilibrium sequences for certain initial points. However, the solution is defined through a certain abstaining strategy of Player 2 in which he sets his control variable at zero and leaves the problem to Player 1. Thus the resulting solution of the well-known time optimal problem is shown in Figure 4.4.

The solution seems biased to Player 1 since while he is interested in time only, Player 2 is interested in a performance index which has a weightage for the first player's fuel also! Finally since each player is interested in the termination of the game, if only one player has observations, then he has to follow his opponent's preferences.

4.5 CONCLUSIONS

In this chapter, the various switching surfaces encountered in optimal control and differential game problems are classified and general construction procedures are indicated. Though at present the study of these surfaces is mostly motivated by examples, we concur with Isaacs (1969) that a comprehensive theory of differential games can be developed through a thorough study of the switching surfaces.

The complete noncooperative solution of the double integral plant problem is presented. In the process, we come across many features which are met with in finite games such as multiplicity, nonequivalence and noninterchangeability of equilibrium points, application of Payoff and Risk Dominance concepts for defining solutions etc. However the nonplayability of recombined strategies as we saw in Section 4.4, consequent to the differential game having different Normal Forms has no parallel in finite games.

An important method of obtaining solution, viz., by numerical computation, is very difficult in these problems because of the dimensionality of the problem, nonavailability of reliable and efficient computational methods in finite games and because of each player's cost at the equilibrium point being insensitive to his own strategy but being sensitive to the

other player's strategies (see Starr 1969). Above all the noncooperative solution requires a selection of the equilibrium sequences which are sometimes uncountable.

In the next chapter we consider Cooperative Solutions of differential games.

CHAPTER V

COOPERATIVE SOLUTIONS AND MULTICRITERION OPTIMAL CONTROL

5.1 INTRODUCTION

In Chapter II we observed that Pareto optimality is a weak optimality concept and that it is the central theme of all Cooperative Solutions. This chapter deals with the cooperative solutions of N-person differential games and the application of the results developed in this thesis to Multicriterion Optimal Control.

The necessary conditions for Pareto optimality derived by Chang (1966) and Das and Sharma (1969) are stated in Section 5.2. This is followed by the cooperative solutions of differential games. We discuss the theory of multicriterion optimal control problems in Section 5.4. Specifically we show that multicriterion optimal control problems can be solved as cooperative N-person differential games, with equal information to all the players.

We describe in Section 5.5 a problem in optimal control giving the sensitivity of optimal control for small changes in performance index and adapt this as a computational method of obtaining the solution of multicriterion optimal control problems. In Section 5.6 we complete the Bicriterion Optimal Control Problem of minimizing the time and fuel of a double integral plant carried throughout the thesis.

5.2 PARETO OPTIMALITY CONCEPT

We recall the deterministic differential game formulated in Section 2.3. The game satisfies the state equation

$$\dot{x} = f(x, \underline{u}, t) \tag{5.1}$$

where x and $\underline{u}$ are of dimensions n and r respectively[1]. The initial condition at time t_o and the terminal surface are specified as

$$x(t_o) = x_o \tag{5.2}$$

$$\psi(x_f, t_f) = 0 \tag{5.3}$$

or alternatively as

$$t_f = T(\sigma) \quad ; \quad x_f = X(\sigma) \tag{5.4}$$

The p^{th} player chooses his control action $u^p \in \Omega^p$, so as to minimize his payoff functional

$$J^p[x_o, t_o, \underline{u}] = \phi^p(x_f, t_f) + \int_{t_o}^{t_f} L^p(x, \underline{u}, t)dt \tag{5.5}$$

The payoff functional vector $\underline{J} = (J^1, \ldots J^N)$ introduces a partial ordering (denoted by $\leq$ below) on the admissible joint control actions $\underline{u} = (u^1, \ldots u^N)$ of the players. According to this ordering, we have for any two

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1 It is obvious that $r = r^1 + \ldots r^N$.

joint control action vectors $\underline{u}_1$ and $\underline{u}_2$

$$\underline{u}_1 \lesssim \underline{u}_2 \quad \text{if and only if} \quad \{J[x_o, t_o, \underline{u}_1] \leqslant J[x_o, t_o, \underline{u}_2]\} \quad (5.6)$$

A vector is said to be Less Than or Equal To another (as in the right-hand side of (5.6))if and only if each component of the first is less than or equal to the corresponding component of the other[2]. We also say that the first vector is Below the second vector if all these are represented in a proper vector space. Thus

$$\{\underline{J}[x_o, t_o, \underline{u}_1] \leqslant \underline{J}[x_o, t_o, \underline{u}_2]\} \text{ if and only if}$$

$$\left\{ \begin{array}{c} J^p[x_o, t_o, \underline{u}_1] \leqslant J^p[x_o, t_o, \underline{u}_2] \\ p = 1, \ldots N \end{array} \right\} \qquad (5.7)^3$$

A control vector $\underline{u}^o$ is Pareto Optimal if there exists no other admissible control action which yields a payoff vector Less Than $\underline{J}[x_o, t_o, \underline{u}^o] = \underline{V}(x_o, t_o)$, where $\underline{V}$ is called the Value Function. Thus for any $\underline{u}$, we have

$$\underline{J}[x_o, t_o, \underline{u}] \leqslant \underline{V}(x_o, t_o) \quad \text{only if} \quad \underline{J}[x_o, t_o, \underline{u}] = \underline{V}(x_o, t_o) \quad (5.8)$$

Thus $\underline{u}^o$ is weakly optimal with respect to the partial ordering defined by (5.6). The necessary conditions for $\underline{u}^o$ to be Pareto optimal are stated below.

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2 This applies equally well to the relations Not Less Than, Less Than, Equal To, Strictly Greater Than and Greater Than or Equal To. Also it should be noted that Not Less Than is not the same as Greater Than or Equal To.

3 Thus this in turn defines a partial ordering on the $\underline{J}$ space.

In order that $\underline{u}^o$ and the corresponding trajectory x^o to be Pareto optimal, it is necessary that there exist a nonzero absolutely continuous vector function $(\underline{k}, \lambda(t))^4 = (k^1, \ldots k^N, \lambda_1(t), \ldots \lambda_n(t))$ with the constant vector[4] $\underline{k} \geqslant 0$ such that the following are satisfied:

(i) Euler-Lagrange equations: $x^o(t)$ and $\lambda(t)$ are a solution to the canonical system

$$\dot{x}^o = \frac{\partial H}{\partial \lambda}(x^o, \lambda, u^o, t) \tag{5.9}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x}(x^o, \lambda, u^o, t) \tag{5.10}$$

satisfying the boundary conditions

$$x^o(t_o) = x_o \tag{5.11}$$

$$\psi(x^o(t_f), t_f) = 0 \tag{5.12}$$

where the Hamiltonian function H is given by

$$H(x, \lambda, u, t) = \langle \underline{k}, L(x,u,t)\rangle + \langle \lambda, f(x,u,t)\rangle \tag{5.13}$$

(ii) Transversality conditions: At the terminal time t_f

$$\lambda(t_f) = \frac{\partial[\ \langle \underline{k}, \emptyset(x_f^o, t_f)\rangle + \langle \vartheta, (x_f^o, t_f)\rangle\ ]}{\partial x_f} \tag{5.14}$$

$$H(t_f) = -\frac{\partial[\ \langle \underline{k}, \emptyset(x_f^o, t_f)\rangle + \langle \vartheta, (x_f^o, t_f)\rangle\ ]}{\partial t_f} \tag{5.15}$$

where ϑ is some constant vector. Alternatively,

$$\langle \underline{k}, \emptyset_\sigma\rangle + H(t_f)\ T_\sigma - \lambda(t_f)\ X_\sigma = 0 \tag{5.16}$$

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4 It is to be noted that $\underline{k}$ is similar to λ_o in Pontryagin's Minimum Principle.

(iii) Minimum principle: The function $H(x^o, \lambda, u, t)$ has an absolute minimum as a function of $\underline{u}$ over Ω at $\underline{u} = \underline{u}^o(t)$ for t in $[t_o, t_f]$, i.e.,

$$H(x^o, \lambda, \underline{u}^o, t) = \min_{\underline{u} \in \Omega} H(x^o, \lambda, \underline{u}, t) \tag{5.17}$$

Since if $\underline{k} > 0$, we can choose $\underline{k}$ such that

$$k^1 + k^2 + \ldots k^N = 1 \tag{5.18}$$

it follows that for different Pareto optimal points, we are minimizing different convex combinations of the payoff functionals[5]. The significance of these necessary conditions is this important scalarization of the vector functional problem. It is clear from (5.13) and (5.17) that H and $\underline{u}^o$ and hence the Value Function $\underline{V}$ are functions of $\underline{k}$.

The above results are derived by Chang (1966). Below, we consider the time-invariant Lagrange problem, i.e., f and L^p are not functions of time explicitly and for $p = 1, \ldots N$

$$\phi^p = 0 \tag{5.19}$$

By the known equivalence of this problem and the problem (5.1)-(5.5), the stated results hold.

5 A problem with any particular convex combination of the functionals may not correspond to any Pareto optimal point inasmuch as the stated conditions are necessary only.

The proof given by Chang (1966) is along the same lines as the one given by Pontryagin (1962). One can also construct a heuristic proof similar to the one given by Athans and Falb (1966). We shall only indicate the changes to be made to the proof in (Athans and Falb 1966).

In the present case, we have to define an auxiliary variable z^p for each payoff functional J^p and a vector such that

$$\dot{z}^p = L(x,\underline{u})$$
$$z^p(0) = 0 \qquad (5.20)$$

since we are considering the time-invariant problem. Also let

$$q = (z^1, \ldots z^N, x)$$
$$q = (\underline{z}, x) \qquad (5.21)$$

Interpreted in this context, the principle of optimality implies that any trajectory $\hat{q}$ in the cost-state space cannot be Below (not same as Above because of Footnote 2) the optimal trajectory q^o, i.e., for any $\hat{q} = (\hat{z}, \hat{x})$ such that

$$x = x^o \qquad (5.22)$$

it follows that

$$\hat{z} \nless z^o \qquad (5.23)$$

By the usual temporal and spatial variations of control, the Terminal Cone is constructed and it similarly follows that the Cost Orthant (set product of the cost half-rays) given by

$$\mathcal{C} = \left\{ q : q = (\underline{z},\ x^{o}) \quad ; \quad \underline{z} \leqslant z^{o} \right\} \tag{5.24}$$

does not have an intersection with the interior of the Terminal Cone. Thus the existence of the Separating Hyperplane along with the adjoint equations gives the minimum principle. Transversality conditions also are obtained in a similar manner.

The questions related to existence of Pareto optimal solutions are considered by Olech (1967) and Das and Sharma (1969).

5.3 COOPERATIVE SOLUTIONS OF DIFFERENTIAL GAMES

Some typical examples of Cooperative Differential Games arise in situations involving Collision Avoidance of approaching Aircraft and Naval vessels and Rendezvous of two Spaceships (Wong 1967). In these problems both the players have the same objective functional and can be considered as agents of the same controlling agency. Thus in terms of solution, they are no different from the optimal control problems. The multipursuer-single evader game suggested by Isaacs can be reduced to a two-person

zero-sum game on similar lines since all the pursuers have the same objective of capturing the evader in minimum time by joint effort.

However, in general the payoff functionals of the players may be all different as is indicated in the formulation of the problem in this Thesis. In such a case, the solution is given by Pareto optimal points. Since these points are nonunique Supercriteria are required to solve the game. The various cooperative solutions studied as arbitration schemes differ in terms of the Supercriteria involved. The resulting solution should reflect the strategic potentialities of the players effectively. These include the threat capabilities, powers of forming coalitions etc. which are reflected in the noncooperative play of the game, which is thus necessary in some form for the supercriteria.

As we observed in Chapter II, the solution of a game in which comparison of payoffs and sidepayments between the players are permitted, is given by that Pareto optimal strategy which minimizes the Total Payoff expressed in the common unit of comparison. Thus once again the problem is reduced to an optimal control problem. The distribution of the total payoff among the players is termed the Redistribution Problem and is solved by the

Characteristic Function Theory. The main concept of the theory is the Characteristic Function itself defined for any subset of players (termed Coalition) in the game. It is given by the Security Level of this Coalition (considered as one player) in a two-person zero-sum game with the rest of the players forming the opponent and the payoff given by the sum of the payoffs of the individual players in the Coalition. Thus the extension of these concepts to the present setting is straightforward. For example, for the double integral plant considered in Example 3.2 and Section 4.4, the above consideration reduces to one of finding out who is the stronger player of the two, i.e., whether $c > 1$ or $c < 1$. We will not consider this aspect any further.

The solution of the game without sidepayments is given as the Nash Cooperative Solution[6] (Nash 1953 and Harsanyi 1959). In this case the Dilemma is resolved by considering the Bargaining between the players with the initial point as the noncooperative solution. The resulting solution has many desirable properties such as symmetry, independence of irrelevant alternatives and invariance with respect to utility transformations and reflects the optimal threats and demands of the players.

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6 Nash gave the solution to two-player games. The extension to N players is due to Harsanyi.

The underlying Supercriterion is defined as

$$I(x_o,t_o,\underline{k}) = - \prod_{p=1}^{N} \{ W^p(x_o,t_o) - J^p[x_o,t_o,\underline{u}^o(\underline{k})] \}$$

$$= - \prod_{p=1}^{N} [W^p(x_o,t_o) - V^p(x_o,t_o,\underline{k})] \qquad (5.25)$$

with the terms in the product of (5.25) corresponding to all the players are positive. The solution $\underline{u}^o(\underline{k}^o)$ is Pareto optimal and minimizes the above Supercriterion for all values of $\underline{k}$, i.e.,

$$\min_{\underline{k}} I(x_o,t_o,\underline{k}) = I(x_o,t_o,\underline{k}^o) \qquad (5.26)$$

It is easily seen that the minimization can be performed over all possible values of $\underline{k}$ because all the terms in the product on the right-hand side of (5.25) are positive (see Footnote 5).

5.4 MULTICRITERION OPTIMAL CONTROL PROBLEMS

In this section, we study the formulation and solution concepts of Multicriterion Optimal Control Problems. The formulation of Multicriterion Optimal Control Problems follows the same pattern as the classical optimal control problems except that there are multiple criteria in this case.

The state of the system x of dimension n satisfies the vector differential equation

$$\dot{x} = f(x,u,t) \qquad (5.27)$$

where u the r - dimensional control action vector is restricted such that

$$u \in \Omega \tag{5.28}$$

The state of the system has to be transferred from an initial state $x(t_o) = x_o$ to the terminal surface given by

$$\psi(x_f, t_f) = 0 \tag{5.29}$$

The control law is to be chosen so as to minimize the following criteria for $p = 1, \ldots N$.

$$J^p[x_o, t_o, u] = \phi^p(x_f, t_f) + \int_{t_o}^{t_f} L^p(x, u, t)\, dt \tag{5.30}$$

Also the measurable system outputs may be given by the observation equation

$$y = h(x, t) \tag{5.31}$$

Because of the presence of more than one criterion, the performance of a control law in relation to another may be better with respect to one criterion and worse with respect to another. Expressed mathematically, a vector criterion induces only a partial ordering on the set of control policies while a scalar criterion induces a total ordering.

A control law, with the property that the system performance cannot be improved with respect to any of the

criteria without simultaneously deteriorating the performance with respect to some other, is a basic concept of solution. It is called Noninferior because of its weak optimality property with respect to the partial ordering. It is clear that Noninferiority is equivalent to Pareto Optimality in Section 5.2. Also we saw that the control laws with this property are nonunique and this is the Dilemma introduced by the partial ordering. Without further knowledge (or alternatively without the application of Supercriteria) there is no way of sifting through the various Noninferior control laws to obtain an acceptable solution which should exhibit the tradeoff factors between the criteria effectively. We consider below a few methods suggested in literature (Nelson 1964, Chyung 1967 and Athans and Falb 1966) as such Supercriteria.

One of the methods due to Nelson (1964), specifies some acceptable bounds on all but one criterion and then optimizes the system performance with respect to this free criterion. In essence, the problem is reduced to an optimal control problem with several isoperimetric constraints (Lee 1966). The classical design techniques patently follow this idea, in which it is usual to put bounds on all the criteria such as phase and gain margins, rise time etc. and any solution satisfying these bounds is taken as satisfactory. Thus there is no optimization involved here.

A second method is to order the criteria according to their importance and apply them in a heirarchy (Chyung 1967). Thus, if the optimal controls resulting from the application of the most preferred criterion are nonunique, then they are tested with the second criterion etc. until the resulting control is unique. This method does not ensure the participation of all the criteria in the final selection of the resulting optimal control law. Such method is suited and resorted to in the Chebychev's problem (Johnson 1967) and fuel optimal problem (Athans and Falb 1966) where the optimal controls with respect to the first criterion are grossly nonunique.

A third method suggested in literature (Athans and Falb 1966) is to weigh all the criteria with positive weights and a control law is selected optimal with respect to the weighted index. However many problems may not permit the assumptions in the above methods.

In this thesis, we treat the multicriterion optimal control problems under the framework of N-person differential games (also see Sarma and Prasad 1969). We assume that the control resources represented by u in the formulation (5.27) - (5.31) can be allocated to the various criteria. As we pointed in Chapter I, this may be naturally given. Insofar as the performance criteria are a mathematical characterization of some of the

objectives, we believe that the plant chosen will usually allow such an allocation.

Thus a multicriterion optimal control problem after such an allocation is similar in form to a differential game without sidepayments and with as many players as there are criteria. The addition restriction is that the observations y in (5.31) are the same for all the players. It is for convenience of solution that the Designer breaks the problem and casts it into a game with one player corresponding to each criterion. The solution obtained in this case should more truly reflect the tradeoffs between the indices.

5.5 A COMPUTATIONAL METHOD FOR NASH SOLUTION

In this section, first we present a problem in optimal control giving the sensitivity of the optimal cost and optimal control for small changes in the performance index. For simplicity we assume a fixed time, free end point problem and the control to be unconstrained.

The system equations and the performance index are given as

$$\dot{x} = f(x,u,t) \tag{5.32}$$

$$J[x_o, t_o, u, \epsilon] = \phi(x_f) + \epsilon\phi'(x_f) + \int_{t_o}^{t_f} \left\{ L(x,u,t) + \epsilon L'(x,u,t) \right\} dt \tag{5.33}$$

The optimal control u^o which minimizes (5.33) is a function of ϵ and the resulting optimal Value is given by

$$V(x_o, t_o, \epsilon) = J[x_o, t_o, u^o(\epsilon), \epsilon] \tag{5.34}$$

The optimal control for the case $\epsilon = 0$ is given for the above problem. The optimal control and the optimal cost are to be obtained for any small nonzero ϵ. Now to a first order approximation, since ϵ is small, we have

$$u^o(\epsilon) = u^o(0) + \frac{\partial u^o}{\partial \epsilon} \cdot \epsilon \tag{5.35}$$

and

$$V(\epsilon) = V(0) + \frac{\partial V}{\partial \epsilon} \cdot \epsilon \tag{5.36}$$

Hence we need determine expressions for $\frac{\partial u^o}{\partial \epsilon}$ and $\frac{\partial V}{\partial \epsilon}$ for the above problem.

The optimal Hamiltonian and the canonical equations for the problem are given below. It is obvious that the canonical variables are functions of ϵ.

$$\begin{aligned} H(\epsilon) &= \lambda^T f(x^o, u^o, t) + L(x^o, u^o, t) + \epsilon L'(x^o, u^o, t) \\ &= H(0) + \epsilon \frac{\partial H}{\partial \epsilon} \end{aligned} \tag{5.37}$$

$$\dot{x}^o = f(x^o, u^o, t) \quad ; \quad x^o(t_o) = x_o \tag{5.38}$$

$$\dot{\lambda} = -\frac{\partial H(\epsilon)}{\partial x} \quad ; \quad \lambda(t_f) = \frac{\partial \phi}{\partial x_f} + \epsilon \frac{\partial \phi'}{\partial x_f} \tag{5.39}$$

Thus writing the expressions for $\frac{\partial x^o}{\partial \epsilon}$ and $\frac{\partial \lambda}{\partial \epsilon}$, we have

$$\left(\frac{\dot{\partial x^o}}{\partial \epsilon}\right) = \frac{\partial f}{\partial x^o}\cdot\frac{\partial x^o}{\partial \epsilon} + \frac{\partial f}{\partial u^o}\cdot\frac{\partial u^o}{\partial \epsilon} \quad ; \quad \frac{\partial x^o(t_o)}{\partial \epsilon} = 0 \tag{5.39}$$

$$\left(\frac{\dot{\partial \lambda}}{\partial \epsilon}\right) = -\frac{\partial^2 H(0)}{\partial x^{o2}}\cdot\frac{\partial x^o}{\partial \epsilon} - \frac{\partial^2 H(0)}{\partial x^o \partial u^o}\cdot\frac{\partial u^o}{\partial \epsilon} - \frac{\partial L'}{\partial x}$$

$$\frac{\partial \lambda}{\partial \epsilon}(t_f) = \frac{\partial^2 \phi}{\partial x_f^2}\frac{\partial x^2}{\partial \epsilon}(t_f) + \frac{\partial \phi'}{\partial x_f} \tag{5.40}$$

where all the quantities in (5.39) and (5.40) are evaluated at $\epsilon = 0$. For optimality of $u^o(\epsilon)$ we have

$$\frac{\partial H(\epsilon)}{\partial u^o} = \frac{\partial H(0)}{\partial u^o} + \frac{\partial^2 H(0)}{\partial u^{o2}}\cdot\frac{\partial u^o}{\partial \epsilon}\epsilon + \frac{\partial^2 H(0)}{\partial u^o \partial x^o}\cdot\frac{\partial x^o}{\partial \epsilon}\epsilon$$

$$+ \frac{\partial^2 H(0)}{\partial u^o \partial \lambda}\cdot\frac{\partial \lambda}{\partial \epsilon}\epsilon + \frac{\partial L'}{\partial u^o}\epsilon = 0 \tag{5.41}$$

where the expressions in (5.41) are evaluated corresponding to the optimal control. Taking the partial derivative with respect to ϵ, (5.41) yields

$$\frac{\partial^2 H(\epsilon)}{\partial u^o \partial \epsilon} = \frac{\partial^2 H(0)}{\partial u^{o2}}\cdot\frac{\partial u^o}{\partial \epsilon} + \frac{\partial^2 H(0)}{\partial u^o \partial x^o}\cdot\frac{\partial x^o}{\partial \epsilon} + \frac{\partial^2 H(0)}{\partial u^o \partial}\cdot\frac{\partial \lambda}{\partial \epsilon} + \frac{\partial L'}{\partial u^o} = 0$$

or

$$\frac{\partial u^o}{\partial \epsilon} = -\left(\frac{\partial^2 H(0)}{\partial u^{o2}}\right)^{-1}\left[\frac{\partial^2 H(0)}{\partial u^o \partial x^o}\cdot\frac{\partial x^o}{\partial \epsilon} + \frac{\partial f}{\partial u^o}\frac{\partial \lambda}{\partial \epsilon} + \frac{\partial L'}{\partial u^o}\right] \tag{5.42}$$

Now from (5.33), we have

$$\frac{\partial V}{\partial \epsilon} = \frac{\partial \phi}{\partial x_f}\cdot\frac{\partial x_f}{\partial \epsilon} + \phi' + \int_{t_o}^{t_f}\left\{\frac{\partial L}{\partial x^o}\cdot\frac{\partial x^o}{\partial \epsilon} + \frac{\partial L}{\partial u^o}\cdot\frac{\partial u^o}{\partial \epsilon} + L'\right\}dt \tag{5.43}$$

Equations (5.39), (5.40), (5.42) and (5.43) constitute the solution of the problem.

The preceding problem can be adapted to obtain $\underline{k}^o$ in the Nash solution. The main theme of this method is to assume some $\underline{k}$ initially and move suitably to a new value depending upon the gradient of I, the Nash Supercriterion. We show below that this involves the solution of a nonlinear programming problem.

By the partial differentiation of (5.25), we have

$$\frac{\partial I}{\partial k^p} = I\left\{\sum_{\ell} \frac{\partial V^{\ell}}{\partial k^p} \frac{1}{(W^{\ell}-V^{\ell})}\right\} \tag{5.44}$$

We move in the $\underline{k}$ space by a small displacement vector $\underline{\epsilon}$ such that

$$\epsilon^{1^2} + \dots \epsilon^{N^2} \leqslant \delta \quad . \tag{5.45}$$

where δ is a small number and such that

$$\epsilon^1 + \dots \epsilon^N = 0 \tag{5.46}$$

$$-k^p < \epsilon^p < 1 - k^p \quad ; \quad p = 1, \dots N \tag{5.47}$$

to minimize

$$\delta I = \sum_p \epsilon^p \frac{\partial I}{\partial k^p} \tag{5.48}$$

Constraint (5.47) is meant to keep the new k^p also positive. Finding $\underline{\epsilon}$ to minimize (5.48) satisfying (5.45) - (5.47) constitutes the familiar nonlinear programming problem (Hadley 1964).

This problem can be solved by the general methods available to Convex Programming problems such as Gradient Projection method and the method of Feasible Directions. However, the problem being one of minimizing a linear function subject to one quadratic and a number of linear constraints, it can be solved by a special technique (Panne 1966) which terminates in a finite number of iterations compared to the general methods. The method suggested is a combination of the Simplex method and a parametric version of the Simplex and dual methods for quadratic programming (Dantzig 1963). The detailed rules and a simple example are given by Panne (1966). The lack of nonnegativity constraints on the variables can be taken care of by defining auxiliary variables.

After finding $\underline{\epsilon}$ we change $\underline{k}$ and u^o iteratively. Thus after the i^{th} iteration, we have

$$u_{i+1} = u_i^o + \Sigma \frac{\partial u^o}{\partial k_i} \epsilon_i \tag{5.49}$$

$$k_{i+1}^p = k_i^p + \epsilon_i^p \tag{5.50}$$

where subscripts refer to the iteration numbers. Utilizing u_{i+1} as the new guess in the scalarized optimal control problem with the corresponding convex combination represented by $\underline{k}_{i+1}$, we obtain the optimal u_{i+1}^o by a suitable numerical technique. Techniques, which assure convergence

when the assumed control is very near the optimal control, are enough for this purpose.

This iterative procedure can be repeated till we reach the optimal value of $\underline{k}^o$ in the Nash solution along with the corresponding optimal control $u^o(\underline{k}^o)$.

5.6 A BICRITERION OPTIMAL CONTROL PROBLEM

In this section, we apply the results of the earlier sections to an illustrative example. The system is the familiar double integral plant with bounded inputs, i.e.,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u^1 + cu^2 \end{aligned} \tag{5.51}$$

$$|u^1| \leqslant 1 \quad ; \quad |u^2| \leqslant 1 \tag{5.52}$$

The inputs are to be chosen to minimize the criteria

$$\begin{aligned} J^1[x_o,u] &= \int_0^{t_f} dt \\ J^2[x_o,u] &= \int_0^{t_f} \left\{ |u^1| + b|u^2| \right\} dt \end{aligned} \tag{5.53}$$

while driving the state of the system from an arbitrary point x_o at time $t = 0$ to the origin, i.e., $x_1(t_f) = x_2(t_f) = 0$. The terminal time t_f is assumed free.

Such a model might represent the single-axis attitude control of a satellite which is equipped with an

lectric motor as well as reaction jets, for control purposes. hile the batteries driving the electric motor are rechargeable y solar radiation, the gas consumed for the reaction jets annot be replenished. The problem formulated corresponds to maneuver which should be performed in minimum time, consuming inimum fuel. The electric motor is primarily used for achieving he minimum time criterion while the reaction jets are sparingly sed with the objective of total fuel minimization.

As the allocation of the control resources to the riteria is already made in a natural manner, we can get the ptimal solution as the Nash cooperative solution. The non-ooperative solution of this problem, which is already presented n Section 3.4(Example 3.2) and Section 4.4, is necessary for he Nash Supercriterion.

For the different Pareto optimal points, we solve the calarized optimal control problems with the criterion unctionals (parametrized by k') given by

$$[x_0, u, \underline{k}] = \int_0^{t_f} \{ k' + (1-k')(|u^1| + b|u^2|) \} dt \qquad (5.54)$$

or $0 \leq k' \leq 1$.

pplication of the Minimum Principle :

To determine the optimal strategies, we minimize he Hamiltonian H with respect to u^1 and u^2 where

$$H = k' + (1-k')(|u^1| + b|u^2|) + \lambda_1 x_2 + \lambda_2 (u^1 + cu^2) \qquad (5.55)$$

subject to the control variable constraints (5.52) and

$$\dot{\lambda}_1 = 0$$
$$\dot{\lambda}_2 = -\lambda_1 \tag{5.56}$$

Hence we get

$$u^1 = -\operatorname{dez} \frac{\lambda_2}{1-k'}$$
$$u^2 = -\operatorname{dez} \frac{\lambda_2 c}{b(1-k')} \tag{5.57}$$

This problem is a two-input version of the problem given in Athans and Falb (1966). The optimal control law is stated below, the proof of which is indicated in Appendix D.

For the various cases, the optimal control sequences are tabulated in Table 5.1. In particular, when $c = b$ and $k' = 0$, the optimal control does not exist for certain starting points. However, similar to the one-input case discussed by Athans and Falb (1966), ϵ - optimal controls exist. In this and some other cases (for example when $c > b$ and $k' = \frac{c-b}{1-b+c}$), it may also be noted that there are uncountable number of Pareto optimal points corresponding to one scalarized optimal control problem with the same k'.

The switching surfaces $\gamma_{1\epsilon}$ and $\gamma_{\epsilon 1}$ for $0 \leq \epsilon \leq 1$ and Γ_ℓ for $\ell = 0, 1, 2$ are defined below.

$$\gamma_{1\epsilon} = \left\{ (x_1, x_2) : x_1 = -\frac{x_2^2}{2(1+c\epsilon)} \operatorname{sgn} x_2 \right\} \tag{5.58}$$

TABLE 5.1

Optimal Control Sequences

Case	Condition	Sequences
$c > b$	$k' > \frac{c-b}{1-b+c}$	$\begin{bmatrix} \mp 1 & 0 & 0 & 0 & \pm 1 \\ \mp 1 & \mp 1 & 0 & \pm 1 & \pm 1 \end{bmatrix}$
	$k' = \frac{c-b}{1-b+c}$	$\begin{bmatrix} \mp 1 & 0 & 0 & 0 & \pm \epsilon \\ \mp 1 & \mp 1 & 0 & \pm 1 & \pm 1 \end{bmatrix} \quad 0 \leqslant \epsilon \leqslant 1$
	$k' < \frac{c-b}{1-b+c}$	$\begin{bmatrix} \mp 1 & 0 & 0 & 0 \\ \mp 1 & \mp 1 & 0 & \pm 1 \end{bmatrix}$ and $\begin{bmatrix} \pm 1 & 0 \\ \pm 1 & \pm 1 \end{bmatrix}$
$c = b$	$k' = 0$	$\begin{bmatrix} \pm \epsilon_1 \\ \pm \epsilon_2 \end{bmatrix} \quad 0 \leqslant \epsilon_1, \epsilon_2 \leqslant 1$
	$0 < k' < 1$	$\begin{bmatrix} \mp 1 & 0 & \pm 1 \\ \mp 1 & 0 & \pm 1 \end{bmatrix}$
	$k' = 1$	$\begin{bmatrix} \mp 1 & \pm 1 \\ \mp 1 & \pm 1 \end{bmatrix}$
$c < b$	$k' > \frac{b-c}{b}$	$\begin{bmatrix} \mp 1 & \mp 1 & 0 & \pm 1 & \pm 1 \\ \mp 1 & 0 & 0 & 0 & \pm 1 \end{bmatrix}$
	$k' = \frac{b-c}{b}$	$\begin{bmatrix} \mp 1 & \mp 1 & 0 & \pm 1 & \pm 1 \\ \mp 1 & 0 & 0 & 0 & \pm \epsilon \end{bmatrix} \quad 0 \leqslant \epsilon \leqslant 1$
	$k' < \frac{b-c}{b}$	$\begin{bmatrix} \mp 1 & \mp 1 & 0 & \pm 1 \\ \mp 1 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} \pm 1 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}$

$$\gamma_{\epsilon 1} = \left\{(x_1, x_2) \;:\; x_1 = -\frac{x_2^2}{2(\epsilon + c)} \operatorname{sgn} x_2\right\} \tag{5.59}$$

$$\Gamma_\ell = \left\{(x_1, x_2) \;:\; x_1 = -\frac{\alpha_\ell}{2} x_2^2 \operatorname{sgn} x_2\right\} \tag{5.60}$$

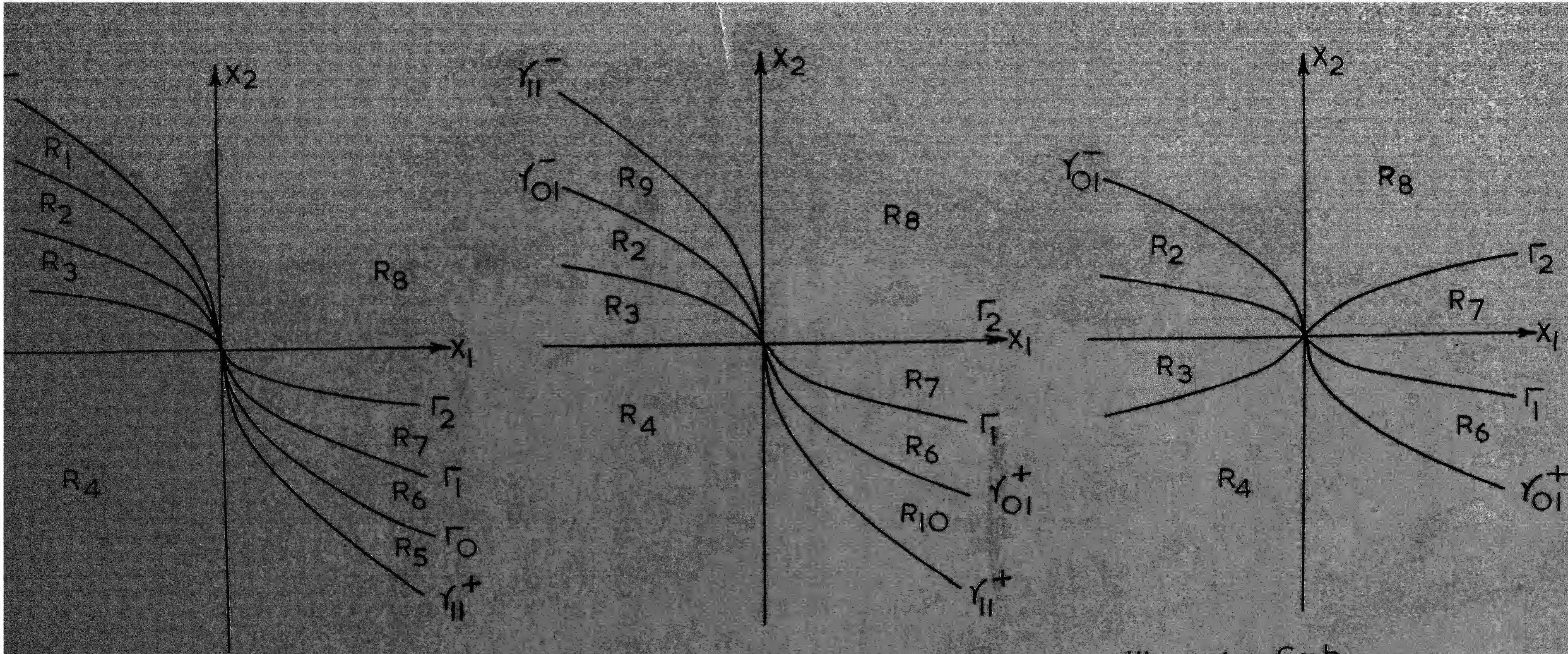
where α_o, α_1 and α_2 are given below for the case $c > b$.

$$\alpha_o = \begin{cases} \dfrac{k'^2 c + 2k'(1-k')(c-b) - (1-k')^2 (b-c)^2}{k'^2 c(1+c)} \;; & k' > \dfrac{c-b}{1-b+c} \\[2ex] \dfrac{1}{c} & k' \leqslant \dfrac{c-b}{1-b+c} \end{cases} \tag{5.61}$$

$$\alpha_1 = \alpha_o + \frac{4(1-k')b}{k'c} \tag{5.62}$$

$$\alpha_2 = \begin{cases} \alpha_1 + \dfrac{(1-k')(c-b)[2k'-(1-k')(c-b)]}{{}'^2 c}\left\{\dfrac{k'^2}{[k'-(c-b)(1-k')]^2}\right\} \\ \qquad\qquad \text{for} \quad k' > \dfrac{c-b}{1-b+c} \\[2ex] \alpha_1 + \dfrac{(1-k')(c-b)[2k'-(1-k')(c-b)]}{{}'^2 c}\left\{\dfrac{-k'^2}{[k'-(c-b)(1-k')]^2}\right\} \\ \qquad\qquad \text{for} \quad k' < \dfrac{c-b}{1-b+c} \end{cases} \tag{5.63}$$

The switching curves for the case $c > b$ are shown in Figure 5.1. The curves Γ_o, Γ_1 and Γ_2 move away from γ_{11} as k' decreases and when $k' = \frac{c-b}{1-b+c}$, Γ_o coincides with γ_{o1} and Γ_2 coincides with the x_1 - axis. As k' decreases further, Γ_2 changes quadrants. According to the



(i) $k' > \frac{c-b}{1-b+c}$ (ii) $k' = \frac{c-b}{1-b+c}$ (iii) $k' < \frac{c-b}{1-b+c}$

FIG. 5.1 SWITCHING SURFACES FOR THE COOPERATIVE SOLUTION CASE $c > b$

TABLE 5.2

Value Function (V^1, V^2) of Cooperative Solution (Case c b)

Region	V^1	V^2
$R_1 \cup R_5$	$\frac{x_2 \operatorname{sgn} x_2}{c} - \frac{1}{c}\sqrt{\frac{x_2^2 + 2x_1 c \operatorname{sgn} x_2}{1+c}}$	$\frac{b x_2 \operatorname{sgn} x_2}{c} + \frac{c-b}{c}\sqrt{\frac{x_2^2 + 2x_1 c \operatorname{sgn} x_2}{1+c}}$
$R_2 \cup R_6$	$-\frac{x_1 + \frac{\alpha_0}{2} x_2^2 \operatorname{sgn} x_2}{x_2} + \frac{x_2 \operatorname{sgn} x_2}{c} \times \left(1 - \sqrt{\frac{1-\alpha_0 c}{1+c}}\right)$	$\frac{b x_2 \operatorname{sgn} x_2}{c} + \frac{(c-b) x_2 \operatorname{sgn} x_2}{c}\sqrt{\frac{1-c\alpha_0}{1+c}}$
R_3	$-\frac{x_2}{c} + (\beta_1^1 + \frac{1}{c})\sqrt{\frac{x_2^2 - 2x_1 c}{1+\alpha_1 c}}$	$-\frac{b x_2}{c} + (\beta_1^2 + \frac{b}{c})\sqrt{\frac{x_2^2 - 2x_1 c}{1+\alpha_1 c}}$
R_7	$\frac{x_2}{c} + (\beta_1^1 + \frac{1}{c})\sqrt{\frac{x_2^2 + 2x_1 c}{1+\alpha_1 c}}$	$\frac{b x_2}{c} + (\beta_1^2 + \frac{b}{c})\sqrt{\frac{x_2^2 + 2x_1 c}{1+\alpha_1 c}}$

Continued

Region	V^1	V^2
R_4	$-\frac{x_2}{1+c} + (\beta_2^1 + \frac{1}{1+c}\operatorname{sgn}\alpha_2)\, X \sqrt{\frac{x_2^2-2x_1(1+c)}{1+\alpha_2(1+c)}}$	$-\frac{(1+b)x_2}{1+c} + (\beta_2^2 + \frac{1+b}{1+c}\operatorname{sgn}\alpha_2)\, X \sqrt{\frac{x_2^2-2x_1(1+c)}{1+\alpha_2(1+c)}}$
R_8	$\frac{x_2}{1+c} + (\beta_2^1 + \frac{\operatorname{sgn}\alpha_2}{1+c})\, X \sqrt{\frac{x_2^2+2x_1(1+c)}{1+\alpha_2(1+c)}}$	$\frac{(1+b)x_2}{1+c} + (\beta_2^2 + \frac{(1+b)\operatorname{sgn}\alpha_2}{1+c})\, X \sqrt{\frac{x_2^2+2x_1(1+c)}{1+\alpha_2(1+c)}}$
$R_9 \cup R_{10}$	Between the limits $\frac{x_2\operatorname{sgn} x_2}{c} - \frac{1}{c}\sqrt{\frac{x_2^2+2x_1c\operatorname{sgn} x_2}{1+c}}$ and $\frac{x_2\operatorname{sgn} x_2}{1+c} + \frac{1}{1+c}\, X \sqrt{\frac{\lvert x_2^2+2x_1(1+c)\operatorname{sgn} x_2\rvert}{c}}$	Between the limits $\frac{bx_2\operatorname{sgn} x_2}{c} + \frac{c-b}{c}\sqrt{\frac{x_2^2+2x_1c\operatorname{sgn} x_2}{1+c}}$ and $\frac{(b+1)x_2\operatorname{sgn} x_2}{1+c} + \frac{b-c}{1+c}\, X \sqrt{\frac{\lvert x_2^2+2x_1(1+c)\operatorname{sgn} x2\rvert}{c}}$

$$\beta_2^1 = -\frac{1}{c}\,\text{sgn}\,\alpha_2 + \sqrt{\frac{1+\alpha_2 c}{1+\alpha_1 c}}\left(\beta_1^1 + \frac{1}{c}\,\text{sgn}\,\alpha_2\right) \tag{5.66}$$

$$\beta_1^2 = \frac{b}{c} + \frac{c-b}{c}\sqrt{\frac{1-c\alpha_0}{1+c}} \tag{5.67}$$

$$\beta_2^2 = -\frac{b}{c}\,\text{sgn}\,\alpha_2 + \sqrt{\frac{1+\alpha_2 c}{1+\alpha_1 c}}\left(\beta_1^2 + \frac{b}{c}\,\text{sgn}\,\alpha_2\right) \tag{5.68}$$

From the entries in Tables 3.1 and 5.1, it is clear that a substitution of the form

$$x_1 = -\frac{\sigma x_2^2}{2}\,\text{sgn}\,x_2 \tag{5.69}$$

parametrizes the state space by σ . This will result in all the Value Functions (W^1, W^2) and (V^1, V^2) proportional to $|x_2|$, the proportionality factors being functions of σ.

For the solution of the Bicriterion Optimal Control Problem for any initial state (x_1, x_2), we have to find k'^{o} such that it minimizes the Nash Supercriterion

$$I(x_1, x_2, k') = -[W^1(x_1,x_2) - V^1(x_1,x_2,k')]\,[W^2(x_1,x_2) - V^2(x_1,x_2,k')] \tag{5.70}$$

Thus we have

$$I(x_1, x_2, k'^{o}) = \min_{k'} I(x_1, x_2, k') \tag{5.71}$$

The minimization in (5.71) is done over the k' corresponding to which $\underline{V}(k')$ dominates $\underline{W}$ (i.e., $\underline{V}(k') \leqslant \underline{W}$).

Once again it is easy to infer from (5.69) that the above minimization depends only on σ and not on (x_1, x_2). Though analytical expressions for k'^{o} in terms of σ, c and b are difficult to obtain, it can be concluded that k'^{o} is constant on a curve given by (5.69). The nature of this curve is the same as the various switching curves shown in Figure 5.1.

Computational Experience with the Example :

The difficulties associated in the numerical computation of the noncooperative solution of differential games are already noted in Section 4.5 (see also Starr 1969). Because of the multisided nature of the optimization involved, the methods applicable to optimal control are not directly applicable to the case of differential games without drastic modifications.

For the case c = b of the example considered, with imperfect information to the players, the Conjugate Gradient Method (Lasdon et.al. 1967) is tried for each player both in alternate iterations and in alternate optimizations. In either case, the free terminal time is determined by satisfying

$$\left.\frac{\partial J^p}{\partial t}\right|_{t_f} = 0 \tag{5.72}$$

The terminal condition of reaching the origin is sought to be met by introducing the Penalty Function $\frac{1}{2} M^p \| x_f \|^2$ and modifying the performance index

$$J^{p'} = J^p + \frac{1}{2} M^p \| x_f \|^2 \qquad (5.73)$$

Thus the problem is converted into a free end point problem. For the constraints on the control variables we follow the modification by Pagurek and Woodside (1968). The instability which arises in this technique can be attributed to the reasons listed in Section 4.5).

For the Nash Cooperative Solution of this problem, the theory presented in Section 5.5 is not directly applicable because the Hamiltonian is linear in the control variables which are bounded. Thus we have

$$H_{uu} = 0 \qquad (5.74)$$

thus making (5.42) inapplicable. However, because there are only two players, the Pareto optimal points are parametrized by a scalar parameter k'. Thus the method is unnecessary. The optimal value k'^o can be determined by a simple search technique such as assuming $k' = 1$ and changing it in small steps in a one-dimensional search to minimize the Nash Supercriterion. Conjugate Gradient Method II (Pagurek and Woodside 1968), which gives good convergence when the guess is close to the optimal one, is used. The resulting optimal control is taken as the guess for the next value of k'.

The procedure involves the solution of several scalarized optimal control problems with different values of the parameter k', with each problem having Penalties to meet the terminal constraints. The convergence and accuracy of the results depend upon the Penalties chosen at each step. The Penalty Function approaches are not well developed for the optimal control problems in contrast to the case of Nonlinear Programming Problems (Fiacco and McCormick 1968). Another salient feature already discussed earlier is the presence of uncountable number of Pareto optimal points corresponding to the same value of k' and all of them being relevant in the minimization of the Super-criterion. These problems are continuing to receive attention.

5.7 CONCLUSIONS

In this chapter, the Pareto optimality concept is discussed in detail along with the cooperative solutions of differential games. The necessary conditions for Pareto optimality are equivalent to a scalarization of the vector minimization.

Multicriterion optimal control problems are solved as N-person differential games without sidepayments and with equal information to all the players. Nash solution is suggested as a solution and a computational procedure is suggested by utilizing a certain sensitivity problem in optimal control.

The solution of the Bicriterion Optimal Control of the double integral plant, some aspects of which are presented in earlier chapters, is completed. In the next chapter, we discuss some problems of current control-theoretic interest in the light of the results in this chapter.

CHAPTER VI

MULTICRITERION OPTIMAL CONTROL UNDER UNCERTAINTY

6.1 INTRODUCTION

In the earlier chapters, we studied N-person differential games and multicriterion optimal control problems under a deterministic framework. We relax this restriction in this chapter on the lines indicated in Chapter II. This enables us to consider some of the current problems of interest in Optimal Control Theory in the light of the results obtained so far in this thesis.

The general formulation in Chapter II gives rise to Stochastic Differential Games with imperfect and incomplete information. A general class of stochastic differential games with perfect information are studied by Kushner and Chamberlain (1969). Markov Positional Games, studied by (Sarma et.al. 1969, Ragade 1968), are stochastic differential games with imperfect information. Finite games with incomplete information appeared recently (Harsanyi 1968).

In Section 6.2, we present the signal design problem for system identification. It will be shown that this problem can be solved as a stochastic optimal control

problem. This is followed by a discussion on Multi-criterion Stochastic Optimal Control. An example of a linear system with two inputs and two quadratic criteria is worked out. Extensions to Adaptive Control - the problem of control under uncertainty - are also indicated in Section 6.3.

The main result in Chapter V, that multicriterion optimal control problems can be solved as N-person differential games without sidepayments but with the added restriction that all the players have equal information, still holds under the stochastic formulation. Thus multicriterion stochastic optimal control problems are easier to solve compared to the general stochastic differential games in which the information to the various players is different. The results presented in this chapter are mainly exploratory in nature.

6.2 SIGNAL DESIGN FOR SYSTEM IDENTIFICATION

Here we consider the signal design problem for the Identification of a system. This problem arose originally in the context of controlling an uncertain plant. The uncertainty might be in terms of some parameters in the plant dynamics or its impulse response in the linear case. In the earlier literature (for a survey see Eveleigh 1967), controllers with adjustable parameters

were suggested with schemes to continuously monitor the controller parameters depending upon the identification system, which is understood in the sense of the unknown parameter estimation. In this connection, there have been attempts at designing separate input signals for the purpose of identification. Now such a problem is described below in the time domain.

We consider a dynamic system whose state x_s and measurable[1] outputs y, of dimensions n_s and m respectively, satisfy a known form of dynamic equations

$$\dot{x}_s = f_s(x_s, x_\sigma, u, w_s, t) \tag{6.1}$$ [2]

$$y = h_s(x_s, w_2, t) \tag{6.2}$$ [2]

where u denotes the r - dimensional vector input to the system, w_s and w_2 are uncorrelated white gaussian noises of dimensions ρ_1 and ρ_2 with means zero and covariances Θ_s and Θ_2 respectively. The set of unknown parameters, which cannot be observed directly, is represented by x_σ. These parameters can as well be in (6.2). Depending upon the statistical characteristics and any markov property satisfied by them, they can be written as

$$\dot{x}_\sigma = f_\sigma(x_\sigma, w_\sigma, t) \tag{6.3}$$ [2]

---

1 The outputs are physically measurable with suitable sensors.

2 The equations (6.1)-(6.3) are linear in the noise terms w_s and w. Their form will usually be different under the rigorous framework of Stochastic Differential Equations.

By appending x_σ with x_s to form a single vector x, one can write (6.1)-(6.3) as

$$\dot{x} = f(x, u, w_1, t) \tag{6.4}$$

$$y = h(x, w_2, t) \tag{6.5}$$

Footnote 2 is valid for (6.4) and (6.5) as well.

For any u, the estimated state trajectory $\hat{x}$ of the system (6.4) and (6.5) includes $\hat{x}_\sigma$ the parameter trajectory. If there is no other control task, the freedom in the choice of u can be exercised in minimizing the criterion functional which depends upon the errors in estimation and the cost of the inputs over the fixed time interval $[t_o, t_f]$ of interest

$$J[u, t_o] = E_x \int_{t_o}^{t_f} \left\{ \| x-\hat{x} \|^2_{Q(t)} + \| u \|^2_{R(t)} \right\} dt \tag{6.6}$$

where

$$\hat{x}(t) = E[x(t) \mid y^t] \tag{6.7}$$

In (6.6), y^t represents the cumulated observations upto time t.

$$y^t = \left\{ y(\tau) : t_o \leq \tau < t \right\} \tag{6.8}$$

Equation (6.5) can also be written as

$$J[u, t_o] = E_{y^{t_o}} E_x [J_{t_o}[u] \mid y^{t_o}] \tag{6.9}$$

where y^{t_o} is a priori information and for any t

in $[t_o, t_f]$,

$$J_\tau[u] = \int_\tau^{t_f} \{\|x-\hat{x}\|^2_{Q(t)} + \|u\|^2_{R(t)}\} dt \qquad (6.10)$$

In (6.6) and (6.10), $Q(t)$ is chosen to suitably weigh only the unknown parameters in the total state.

This problem (6.1)-(6.10) is termed as the signal design problem for system identification. Thus the formulation is similar to the stochastic optimal control problem. By the principle of optimality it follows that $u^o_{[\tau,t_f]}$, a segment of the optimal control u^o for any τ in $[t_o, t_f]$, should minimize

$$J[u,\tau] = E_{y^\tau} E_x [J_\tau[u] \mid y] \qquad (6.11)$$

The solution of the problem $u^o(y^t, t)$ or alternatively $u^o(p(\cdot|y^t), t)$ is given by solving Bellman's Value equation and a modified Chapman-Kolmogorov equation for $p(\cdot|y^t)$ along with the system equations. By averaging the expression for the conditional density $p(\cdot|y^t)$, we get $\hat{x}$. We consider below a simple example, which envelopes the problem studied by Levadi (1966) using the reproducing kernel Hilbert space.

Example 6.1 :

The system is given by the equations

$$\dot{x} = A(u,t)\,x + D_1(t)\,w_1 \qquad (6.12)$$
$$y = C(t)\,x + D_2(t)\,w_2$$

where the state x and the observations y are of dimensions n and m respectively. w_1 and w_2 are uncorrelated white noises of dimensions ρ_1 and ρ_2 with mean zero and of covariances Θ_1 and Θ_2 respectively. The matrices A, C, D_1 and D_2 are of proper dimensions. We obtain the input u, as a program (or open loop control law) independent of x or $\hat{x}$, so as to minimize the performance index

$$J[u,t_o] = E_x \int_{t_o}^{t_f} \|x-\hat{x}\|^2_{Q(t)} + \|u\|^2_{R(t)} \quad dt \tag{6.13}$$

Since the system equations are linear in x and the noises are white gaussian, the optimal filter which gives $\hat{x}$, is of the Kalman type,

$$\dot{\hat{x}} = A(u,t)\,\hat{x} + PC^T\Theta_2^{-1}(y-C\hat{x}) \qquad (6.14)^3$$

where $P = \text{cov}\,(x-\hat{x})$ is the covariance of the error which satisfies the Riccati equation

$$\dot{P} = A^T(u,t)\,P + PA(u,t) - PC^TD_2\Theta_2^{-1}\,D_2^TCP + D_1\Theta_1D_1^T \tag{6.15}$$

with the initial condition $P(t_o) = \text{cov}\,(x(t_o))$.

Now (6.13) can be written as

$$J[u,t_o] = \int_{t_o}^{t_f}\{\langle P,Q\rangle + \langle u,Ru\rangle\}\,dt \tag{6.16}$$

---

3 Transpose notation is used in the examples of this chapter because of the familiarity of the results in this form.

For minimizing (6.16) subject to (6.15), we can apply the Matrix Minimum Principle (Athans 1966), i.e., we have

$$H(P,\Psi,u,t) = \langle P,Q \rangle + \langle u,Ru \rangle + \langle \Psi,\dot{P} \rangle \tag{6.17}$$

$$\dot{\Psi} = -\frac{\partial H^o}{\partial P} \quad ; \quad \Psi(t_f) = 0 \tag{6.18}$$

and

$$H^o = H(P,\Psi, u^o,t) = \min_u H(P,\Psi,u,t) \tag{6.19}$$

Thus u^o is obtained by minimizing the Hamiltonian (6.17). A similar procedure is used by Athans and Schweppe (1967) to solve the problem of synthesizing an optimum modulation signal.

For the more general system in (6.4) and (6.5), the filter equations can be assumed to be the suboptimal filter given by Schwartz (1966). For the suboptimal Schwartz filter, the equation for the error covariance P is an ordinary differential equation. As such, the matrix minimum principle can similarly be applied to determine the optimal input u^o. This problem may be termed as a signal design problem for a system with a constrained state estimator.

6.3 MULTICRITERION OPTIMAL CONTROL WITH UNCERTAINTY

In this section, we discuss the optimal control of an uncertain plant given more than one criterion. We consider the plant (6.1) and (6.2) for this purpose.

The control u is to be chosen as a function of the cumulated observations y^t, to minimize for $p = 1,\ldots N$.

$$J^p[u,t_o] = E_x \left[\phi^p(x_f,t_f) + \int_{t_o}^{t_f} L^p(x,u,t)dt \right] \qquad (6.20)$$

The multiple criteria arise in many a situation, as for example, when one is interested simultaneously in the identification and in the control of the system.

Stochastic Version :

If the statistical characterization of x in (6.1) is complete in the problem, then the plant can be represented by (6.4) and (6.5) with the noise terms appearing linearly. Because of the presence of the noise terms, the resulting problem is termed stochastic. The information is imperfect because the observations cannot yield the exact state vector.

Even though each criterion in (6.20) will be able to introduce a total ordering on the set of control policies, the criterion functional vector will not be able to do so. This is similar to the deterministic case and the solution is given in terms of the noninferior control laws. Once again a noninferior control law u^o on the interval $[t_o,t_f]$ is defined as in (2.24) and (5.10) with respect to $J^p[u,t_o]$ in (6.20).

By the principle of optimality, $u^o_{[t,t_f]}$ for any t in $[t_o, t_f]$ is also noninferior on the subinterval $[t, t_f]$. Under suitable convexity conditions in the cost-vector space for the feasible controls, it may be possible to scalarize the vector criterion. We assume such a scalarization in the example considered below.

The Supercriterion, for selecting one noninferior control as the solution of the problem, depends upon the noncooperative solution of the game. We present below an example to illustrate the ideas involved.

Example 3.2 : We consider a linear system with two quadratic criteria and with noisy observations.

$$\begin{aligned} \dot{x} &= A(t)\,x + B^1(t)\,u^1 + B^2(t)\,u^2 + D_1(t)\,w_1 \\ y &= C(t)\,x + D_2(t)\,w_2 \end{aligned} \tag{6.21}$$

where the state x, observations y and the control variables u^1 and u^2 are of dimensions n, m, r^1 and r^2 respectively. In (6.21), w_1 and w_2 are uncorrelated zero-mean white noises of dimensions ρ_1 and ρ_2 and with covariances Θ_1 and Θ_2. The matrices A, B^1, B^2, C, D_1 and D_2 are of proper dimensions.

The objective is to minimize the performance indices given for $p = 1,2$ as

$$J^p[u, t_o] = E_{y_{t_o}} E_x J^p_{t_o}[u] \tag{6.22}$$

where

$$J_{\tau}^{p}|u| = \|x_f\|^2_{F^p} + \int_{\tau}^{t_f} \left\{ \|x\|^2_{Q(t)} + \|u^1\|^2_{R_1^p(t)} + \|u^2\|^2_{R_2^p(t)} \right\} dt \tag{6.23}$$

The problem formulated is similar to the two-person game formulated by Rhodes (1969) in which he assumes $D_1 = Q = 0$. Rhodes also assumes two different observation equations for the players (see also Rhodes and Luenberger 1969 and Behn and Ho 1968 for the two-person zero-sum version).

The Nash equilibrium solution is obtained by Rhodes when one of the players has either null or perfect observations. We present the solution to the Bicriterion Control Problem where, by assumption, both the players have imperfect but equal observations. Thus for $p = 1,2$ we have

$$u^{p*}(t) = -R_p^{p^{-1}}(t)\, B^{p^T}(t)\, S^p(t)\, \hat{x}(t) \tag{6.24}$$

where

$$\dot{S}^p = -S^pA - A^TS^p - Q^p - \sum_j (S^jB^jR_j^{j^{-1}}R_j^pR_j^{j^{-1}}B^{j^T}S^j$$
$$- S^pB^jR_j^{j^{-1}}B^{j^T}S^j - S^jB^jR_j^{j^{-1}}B^{j^T}S^p) \tag{6.25}$$

$$S^p(t_f) = F^p$$

$$\dot{\hat{x}} = (A - B^1R_1^{1^{-1}}B^{1^T}S^1 - B^2R_2^{2^{-1}}B^{2^T}S^2)\,\hat{x}$$
$$+ PC^T\Theta_2^{-1}(y - C\hat{x}) \tag{6.26}$$

and

$$\dot{P} = A^T P + PA - PC^T D_2 \Theta_2^{-1} D_2^T CP + D_1 \Theta_1 D_1^T \qquad (6.$$

$$P(t_o) = \text{cov}\,(x_o) \qquad (6.27)$$

The proof of this result can be given by showing the optimality of each player's control law in his one-sided optimal control problem. We follow the results of Rhodes (1969) and Rhodes and Luenberger (1969), making use of the Optimal Return Functions for the players. Thus we have, with $S^p(t_f) = F^p$ and $b^p(t_f) = 0$,

$$W^p(x,t) = x^T S^p(t)\, x + b^p(t) \qquad (6.28)$$

The optimal control $u^{p*}(t)$ is obtained as the argument which minimizes

$$[\,(W_t^p + W_x^p\, \dot{x}) \mid y^t\,] \qquad (6.29)$$

Hence for $u^{1*}(t)$, we get

$$u^{1*}(t) = \arg\min E\,[\{x^T \dot{S}^1 x + \dot{b}^1 + 2S^1 x(\,Ax + B^1 u^1 - B^2 R_2^{2^{-1}} B^{2T} S^2 \hat{x}^2\,) + \|x\|^2_{Q^1} + \|u^1\|^2_{R_1^1} + \|u^2\|^2_{R_2^1}\} \mid y^t]$$

$$= E\,[\,-R_1^{1^{-1}}\, B^{1T}\, S^1\, x \mid y^t\,]$$

$$= -R_1^{1^{-1}}\, B^{1T}\, S^1\, \hat{x} \qquad (6.30).$$

The minimum value of (6.29) equated to zero gives the equations satisfied by S^1 and b^1 as

$$\dot{S}^1 = -A^T S^1 - S^1 A + S^1 B^1 R_1^{1^{-1}} B^{1^T} S^1 + S^1 B^2 R_2^{2^{-1}} B^{2^T} S^2$$
$$+ S^2 B^2 R_2^{2^{-1}} B^{2^T} S^1 - Q^1$$
$$- S^2 B^2 R_2^{2^{-1}} R_2^1 R_2^{2^{-1}} B^{2^T} S^2$$
$$\dot{b}^1 = -\mathrm{Tr}[P(2S^1 B^1 R_1^{1^{-1}} B^{1^T} S^1 + S^1 B^2 R_2^{2^{-1}} B^{2^T} S^2 + S^2 B^2 R_2^{2^{-1}} B^{2^T} S^1)] \quad (6.31)$$

The second player's control $u^{2*}(t)$ is similarly proved to be optimal.

The control law (6.24) is intuitively obvious. Along with the control law for the deterministic problem with perfect observations (3.48), it follows that the Separation Theorem (Wonham 1968) is valid for both the players.

The Pareto optimal control laws are obtained by considering the scalarized stochastic optimal control problem given by (6.21) with the performance index

$$J[u, t_o, k'] = k' J^1[u, t_o] + (1-k') J^2[u, t_o] \quad (6.32)$$

Obviously the Separation Theorem is valid for this class of problems and the control law is given by

$$u^{1^o}(k') = - R^{1^{-1}}(k') B^{1^T} S(k')\hat{x}$$
$$u^{2^o}(k') = - R^{2^{-1}}(k') B^{2^T} S(k')\hat{x} \quad (6.33)$$

where

$$\dot{S}(k') = -A^T S(k') - S(k')A - Q(k') + S(k')B^1 R^{1^{-1}}(k')B^{1^T} S(k')$$
$$+ S(k')B^2 R^{2^{-1}}(k')B^{2^T} S(k') \tag{6.34}$$

$$S(t_f, k') = k'F^1 + (1-k')F^2$$

$$Q(k') = k'Q^1 + (1-k')Q^2 \tag{6.35}$$

and for $p = 1,2$

$$R^p(k') = k'R_p^1 + (1-k')R_p^2 \tag{6.36}$$

This can be proved, as in the case of the Nash equilibrium solution, By by assuming a quadratic form for the Optimal Return Function $V(x,t,k')$.

By assuming the Nash equilibrium solution (6.24) as the noncooperative solution, the optimal k'^o corresponding to the Nash cooperative solution is obtained by minimizing the Nash Supercriterion I with respect to k', where

$$I(y^{t_o}, t_o, k') = -[W^1(y^{t_o}, t_o) - V^1(y^{t_o}, t_o, k')][W^2(y^{t_o}, t_o) - V^2(y^{t_o}, t_o, k')] \tag{6.37}$$

Next we consider the case when the parameters x_6 are not statistically characterized completely.

Adaptive Version :

In the earlier literature on the problem of control under uncertainty, a heirarchical splitting of

the problem into Identification and Control is considered essential and the notion of Adaptation is associated with the two attributes of Identification and Learning. However, this splitting of the problem into two levels, similar to the multilevel techniques (Mesarovic 196), is subjective. Further some of the assumptions in the earlier literature are difficult to justify.

In the stochastic version, where the unknown parameters in the system are statistically characterized completely, we saw that the controller requires in general the current probability density of the state vector (including the parameters) and this can be viewed as a Learning Process. In contrast to this situation, if the statistical characterization of x_ζ in (6.1) is incomplete, then even a single criterion will be unable to introduce a total ordering on the set of control policies. The resulting partial ordering is due to the presence of the uncertainty. Only in this case, one is justified in calling the problem Adaptive and a Supercriterion is essential to resolve the dilemma (Sworder 1966). A possible supercriterion is to complete the statistical characterization of x in a worst sense and design a minimax controller.(also see Ragade and Sarma 1967 for the deterministic case). This turns out to be a Bayes or extended Bayes control law (Sworder 1966) and exhibits learning as explained earlier

in the case of stochastic optimal control. Thus the application of Supercriterion for the solution is what distinguishes an Adaptive Control Problem.

In the presence of more than one criterion, the Supercriterion applied for an Adaptive Control Problem should thus reflect the resolution of the dilemmas caused due to the presence of both the uncertainty and the multiple criteria.

The worst-case type of Supercriterion can be obtained for this problem from the Theory of Approachability and Excludability of Blackwell (1956) for finite two-person zero-sum games with the payoffs being vector-valued. Harsanyi (196a) initiated the solution of finite games with incomplete information. These results will be of considerable significance in arriving at a solution to the above problem. This is suggested for further research.

6.4 CONCLUSIONS

The Theory of Stochastic Differential Games for the cases of imperfect and incomplete information to the players is relatively new with very few significant results and holds promise as a potential area for Research. The problem of Multicriterion Optimal Control of an uncertain plant - both the Stochastic and Adaptive versions - are

simpler problems of the above theory with all the players having equal information. These problems can be solved making use of the concepts of Blackwell (1956) and Harsanyi (1968).

CHAPTER VII

CONCLUSIONS

N-Person Differential Games are a class of Infinite Games with a continuum of strategies and a continuum of moves to the players. They can also be considered as multisided generalizations of Optimal Control Problems. The solution concepts of Finite Games are generally applicable to N-Person Differential Games. Thus the solution depends upon the Information Patterns to the players and the level of Cooperation between the players. Differential Games find application in Economics, Warfare and System Design.

Since any practical system design usually requires the satisfaction of several basically different objectives, the study of Multicriterion Optimal Control Problems assumes great importance. It is shown that they can be solved as Cooperative N-Person Differential Games without sidepayments and with equal information to all the players. The solution obtained in this way reflects the Tradeoff Factors between the various criteria in a game-theoretic sense. It is the equal information feature which makes the solution of these problems comparatively easier compared to general N-Person Differential Games.

In this Thesis, a study of N-Person Differential Games is undertaken mainly in a deterministic framework. Their solution shows many similarities with the solutions of Finite Games on the one hand and of Optimal Control Problems on the other. These include existence of multiple Equilibrium Solutions (including an uncountable number of them) as in Finite Games and the application of a Minimum Principle as in Optimal Control. However they also exhibit features unknown in these areas. For example, unlike in optimal control, they show a wide variety of switching surfaces (studied recently in the case of two-person zero-sum games). Though the study of these surfaces at this stage is mainly motivated through examples, a thorough understanding of these surfaces is an essential prerequisite to a comprehensive theory of Differential Games.

Many types of Information Patterns to the players are possible in Differential Games. In this Thesis, the two extremes of null and perfect information are considered. Consideration of partial information to the players, along with introduction of mixed strategies will permit a study of larger classes of games.

The study of games with imperfect and incomplete information to the players is initiated by introducing Multicriterion Optimal Control Problems under uncertainty.

A thorough study of these problems requires extra mathematical concepts such as the recently developed Calculus of Ito. Such concepts are presented for Optimal Control and Two-Person Zero-Sum Games in a recent Thesis by Mandke (1969).

Lastly, the numerical solution of Differential Games brings out certain problems peculiar to them. Some of these problems are suggested for further investigation and are continuing to receive attention.

APPENDIX A

SWITCHING CURVE REACHING THE ORIGIN

For the Bicriterion Differential Game discussed in Chapters III, IV and V, we derive here the equation for the switching curve along which the initial state (s_1, s_2) reaches the origin under the control law (u^1, u^2).

The game satisfies the state equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u^1 + cu^2 \end{aligned} \tag{A.1}$$

The Payoff Functionals J^1 and J^2 for the players are defined as

$$\begin{aligned} J^1[s,u] &= \int_0^{t_f} dt \\ J^2[s,u] &= \int_0^{t_f} \{ |u^1| + b|u^2| \} \, dt \end{aligned} \tag{A.2}$$

On integrating (A.1) with the initial condition $x_1(0) = s_1$, $x_2(0) = s_2$ we have

$$\begin{aligned} x_2 &= s_2 + (u^1 + cu^2)\, t \\ x_1 &= s_1 + s_2 t + \tfrac{1}{2}(u^1 + cu^2)\, t^2 \end{aligned} \tag{A.3}$$

For the system state to reach origin, there must exist some $t_f > 0$ with $x_1(t_f) = 0$, $x_2(t_f) = 0$ in (A.3).

Hence we get

$$t_f = -\frac{s_2}{u^1+cu^2}$$

$$\operatorname{sgn} s_2 = -\operatorname{sgn}(u^1 + cu^2) \tag{A.4}$$

and $s_1 = -\dfrac{s_2^2}{2(u^1+cu^2)}$.

We now define the switching curve $\gamma_{u^1u^2}$ as follows :

$$\gamma_{u^1u^2} = \left\{(s_1, s_2) : \operatorname{sgn} s_2 = -\operatorname{sgn}(u^1+cu^2),\ s_1 = -\frac{s_2^2}{2(u^1+cu^2)}\right\} \tag{A.5}$$

The values of the payoff functionals (A.2) are therefore given by

$$J^1[s,u] = \int_0^{t_f} dt = t_f = -\frac{s_2}{u^1+cu^2}$$

$$J^2[s,u] = \int_0^{t_f}\left\{|u^1| + b|u^2|\right\}dt \tag{A.6}$$

$$= \left\{|u^1| + b|u^2|\right\}t_f = -\frac{|u^1|+b|u^2|\ s_2}{u^1 + cu^2}$$

After obtaining the terminal sequence and the associated equations (A.5) and (A.6), we proceed to construct the noncooperative or cooperative solution of the game by satisfying the required corner conditions.

APPENDIX B

EVALUATION OF VALUE FUNCTIONS

For the Bicriterion Differential Game discussed in Chapters IIII, IV and V, we give here a general result for the evaluation of the Value Functions.

The game satisfies the state equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u^1 + cu^2 \tag{B.1}$$

The Payoff Functionals for the players are given by

$$J^1[x_o,u] = \int_0^{t_f} dt$$
$$J^2[x_o,u] = \int_0^{t_f} \left\{ |u^1| + b|u^2| \right\} dt \tag{B.2}$$

Suppose the equation of the curve Γ in Figure B.1 is given by

$$\Gamma = \left\{ (s_1, s_2) : s_1 = -\frac{\alpha s_2^2}{2} \right. \tag{B.3}$$

and the Value Functions on Γ are assumed as

$$W^1(s_1,s_2) = \beta^1 |s_2|$$
$$W^2(s_1,s_2) = \beta^2 |s_2| \tag{B.4}$$

We find the Value Functions for points in the region below where the optimal control is (u^1,u^2).

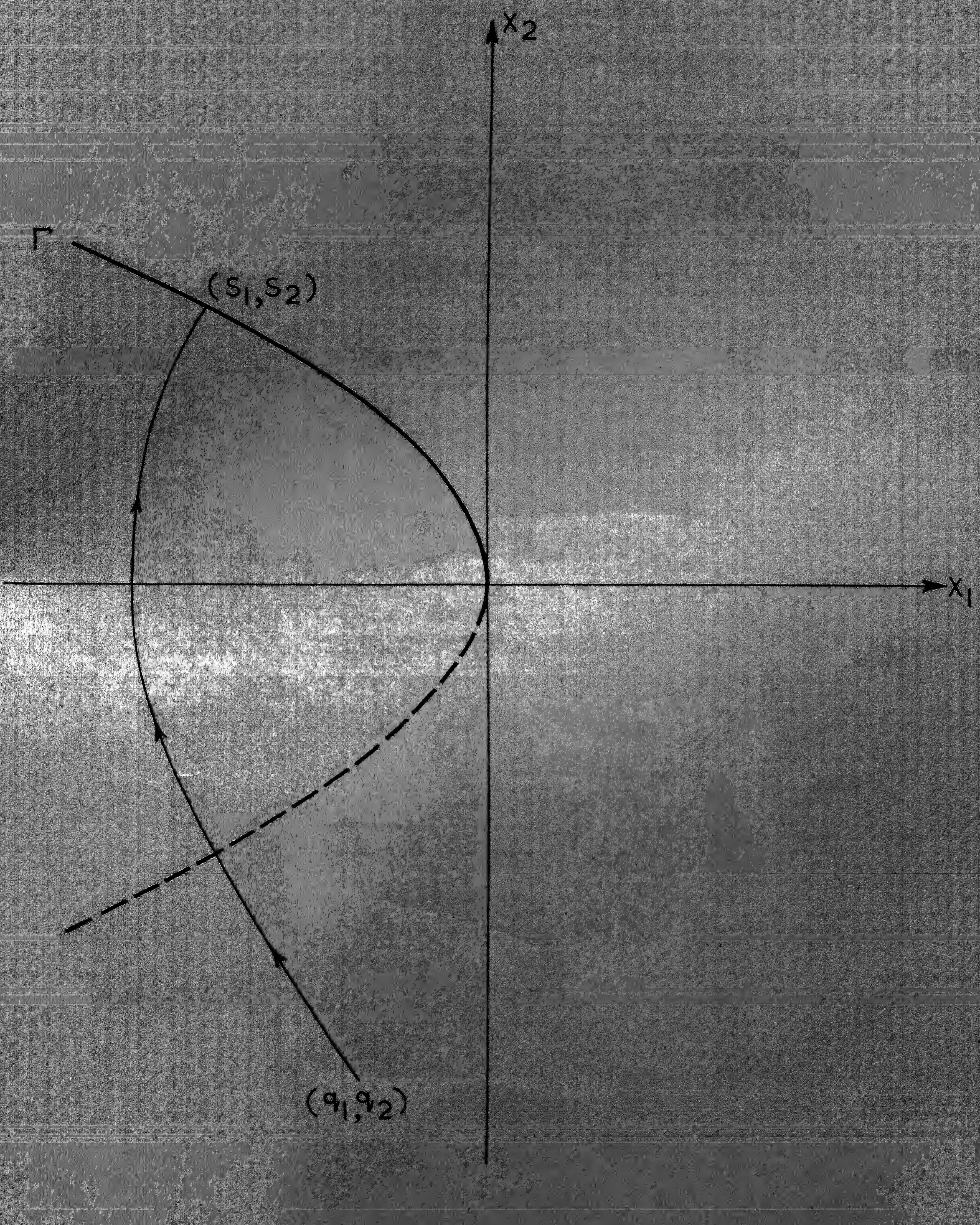


FIG. B.1 AN EXAMPLE FOR THE EVALUATION OF VALUE FUNCTIONS

A typical optimal trajectory starting from (r_1, r_2) and reaching Γ in (s_1, s_2) is shown in Figure B.1 and let t be the time taken. Then solving (B.1) we have

$$s_2 = r_2 + (u^1 + cu^2)\, t$$
$$s_1 = r_1 + r_2 t + \frac{1}{2}(u^1 + cu^2)\, t^2 \tag{B.5}$$

Since (s_1, s_2) lies on Γ, we get

$$r_1 + r_2 t + \frac{1}{2}(u^1+cu^2)\, t^2 = -\frac{\alpha}{2}\left[r_2 + (u^1+cu^2)\, t\right]^2$$

or

$$t^2 \frac{(u^1+cu^2)}{2}\left[1+\alpha(u^1+cu^2)\right] + t\, r_2\left[1+\alpha(u^1+cu^2)\right] + \left(r_1 - \frac{\alpha\, r_2^2}{2}\right) = 0 \tag{B.6}$$

Solving (B.6) for t, we get

$$t = -\frac{r_2}{u^1+cu^2} \pm \frac{1}{u^1+cu^2}\sqrt{\frac{r_2^2 - 2r_1(u^1+cu^2)}{1+\alpha(u^1+cu^2)}} \tag{B.7}$$

where the second value corresponds to the dotted curve symmetrical with respect to Γ.

Substituting (B.7) in (B.5), we have

$$s_2 = \pm\sqrt{\frac{r_2^2 - 2r_1(u^1+cu^2)}{1+\alpha(u^1+cu^2)}} \tag{B.8}$$

Once again the second value corresponds to the dotted curve in Figure B.1. Hence from (B.4) and (B.8) the

Value Functions are given by

$$W^1(r_1,r_2) = -\frac{r_2}{u^1+cu^2} + \sqrt{\frac{r_2^2-2r_1(u^1+cu^2)}{1+\alpha(u^1+cu^2)}}\left\{\beta^1 \pm \frac{1}{u^1+cu^2}\right\}$$

$$W^2(r_1,r_2) = -\frac{r_2(|u^1|+b|u^2|)}{u^1+cu^2} + \sqrt{\frac{r_2^2-2r_1(u^1+cu^2)}{1+\alpha(u^1+cu^2)}}\left[\beta^2 \pm \frac{|u^1|+b|u^2|}{u^1+cu^2}\right] \tag{B.9}$$

Equation (B.9) giving the Value Functions for the trajectories reaching Γ and its symmetric counterpart in Figure B.1 enables us to determine the Noncooperative and Cooperative Value Functions of the game.

APPENDIX C

EQUILIBRIUM SEQUENCES FOR THE NONCOOPERATIVE SOLUTION

Here we obtain the Nash equilibrium sequences for the Bicriterion Differential Game discussed in Chapter IV. The game satisfies the differential equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u^1 + cu^2 \end{aligned} \tag{C.1}$$

The Payoff Functionals for the players are given by

$$\begin{aligned} J^1[x_o,\underline{u}] &= \int_0^{t_f} dt \\ J^2[x_o,\underline{u}] &= \int_0^{t_f} \left\{ |u^1| + b|u^2| \right\} dt \end{aligned} \tag{C.2}$$

The application of the transversality conditions and the minimum principle was shown in Example 3.2. The resulting equations are (3.75), (3.76), (3.66) and (3.71). By making use of these equations, we obtain the terminal sequences for the various cases $c > b$, $c = b$ and $c < b$ that arise in the problem.

Thus for example if we assume the terminal sequence as

$$u^1 = -1 \quad ; \quad u^2 = 0 \tag{C.3}$$

We have from (3.76),

$$\lambda_2^2(t_f) = \lambda_2^1(t_f) = 1. \tag{C.4}$$

From (3.66), (C.4) and (C.3), it is required that

$$1 < \frac{b}{c} \quad \text{or} \quad c < b \tag{C.5}$$

Thus for the sequence $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ (and similarly for the sequence $\begin{bmatrix} +1 \\ 0 \end{bmatrix}$) to be an equilibrium sequence, $c < b$ is necessary.

Similarly, the terminal sequences $\begin{bmatrix} \pm1 \\ \pm1 \end{bmatrix}$ and $\begin{bmatrix} \pm1 \\ \pm\epsilon \end{bmatrix}$ were shown to be optimal for the cases $c > b$ and $c = b$ in Examples 3.2 and 4.1 respectively. Starting from these terminal sequences, we should construct the equilibrium sequences by satisfying the corner conditions at the appropriate switching surfaces. The procedure is similar to that shown in Example 3.2 where this was worked out for the case $c > b$. We present below only the remaining cases.

Case (1) : $c < b$

For this case, the sequences $\begin{bmatrix} \pm1 & \pm1 & \mp1 \\ \pm1 & 0 & 0 \end{bmatrix}$ are in equilibrium provided the condition $2c-b^2+2bc+c^2 > 0$ is also satisfied. When this condition is not satisfied, the equilibrium sequences are $\begin{bmatrix} \pm1 & \mp1 \\ 0 & 0 \end{bmatrix}$. We prove this for the sequences given by the upper values. Figure C.1(i) shows typical plots of $\lambda_2^1, \lambda_2^2, u^1$ and u^2 for this case. The corresponding switching curves, shown in Figure 4.3(i), are constructed below. The state of the game at t_2 and t_3 is denoted by (r_1, r_2) and (s_1, s_2) respectively.

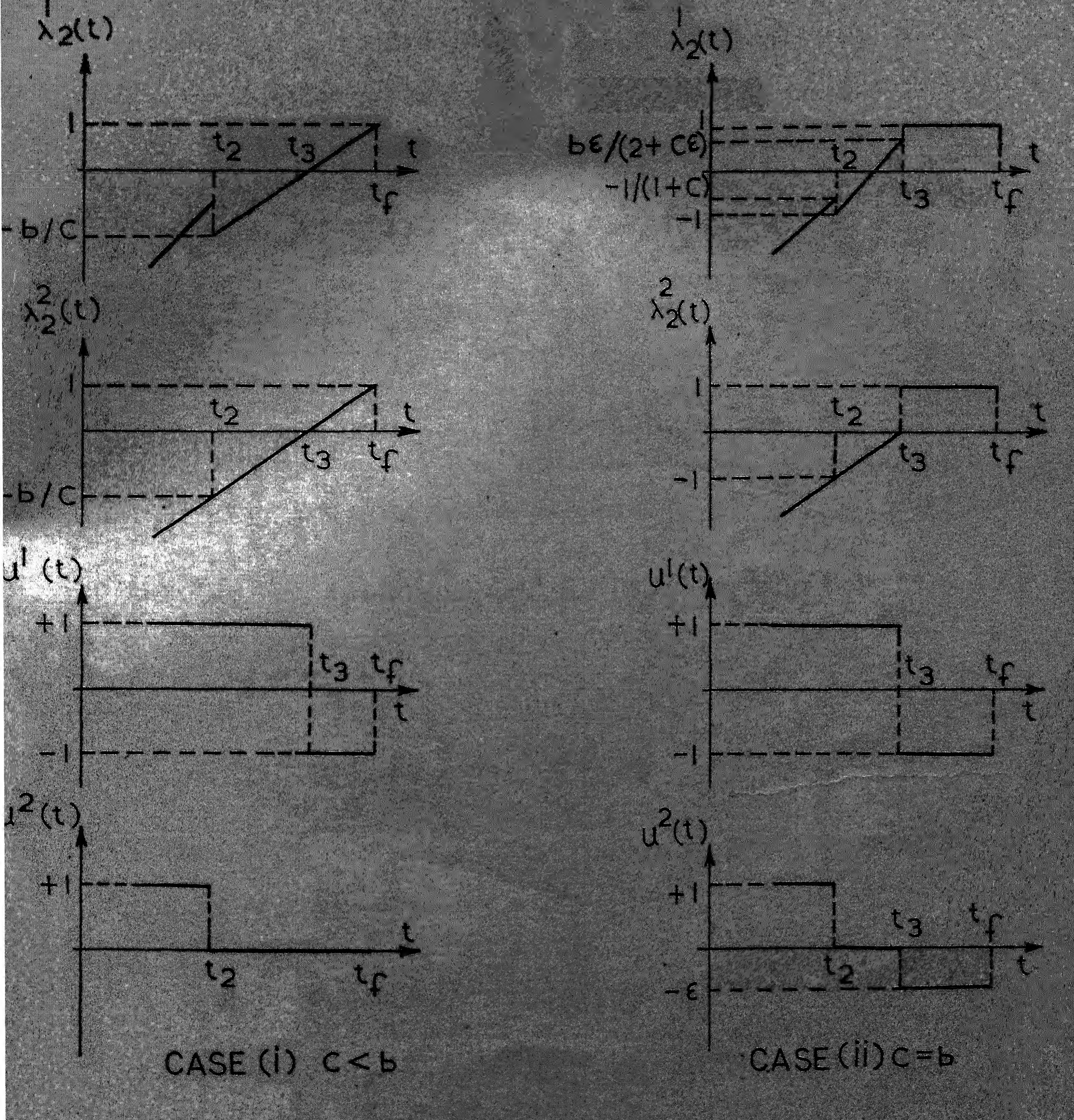


FIG. C.1 RESPONSES OF VARLABLES λ AND u CORRESPONDING TO OPTIMAL STRATEGY FOR THE NONCOOPERATIVE SOLUTION.

Applying the corner conditions at the switching surface γ_{10}^{-} corresponding to the terminal sequence $\begin{bmatrix}-1\\0\end{bmatrix}$, we can easily see that the λ variables are continuous on this switching surface. Thus we have

$$\lambda_1^1(t_2+) = \lambda_1^2(t_2+) = \lambda_1^1(t_3-) = \lambda_1^2(t_3-) = -\frac{1}{s_2}$$
$$\lambda_2^2(t_2+) = \lambda_2^1(t_2+) = -\frac{b}{c} \tag{C.6}$$

Now from (3.71) and (C.6), we have

$$t_3-t_2 = \frac{b}{c}\, s_2 \tag{C.7}$$

Solving the dynamic equations (3.62) we have

$$s_2 = r_2 + \frac{b}{c}\, s_2$$
$$s_1 = r_1 + r_2 \frac{b}{c}\, s_2 + \frac{1}{2}\frac{b^2}{c^2}\, s_2^2 \tag{C.8}$$

Eliminating (s_1, s_2) from (C.8), we get the equation of the switching curve as

$$\Gamma' = \left\{ (r_1, r_2) : r_1 = -\alpha' \frac{r_2^2}{2} \operatorname{sgn} r_2 \right\} \tag{C.9}$$

where

$$\alpha' = \frac{b^2-c^2-2bc}{(c-b)^2} \tag{C.10}$$

Also from (C.6) and (C.8), we have

$$\lambda_1^1(t_2+) = \lambda_1^2(t_2+) = \frac{b-c}{cr_2} \tag{C.11}$$

Applying the corner conditions (3.25) on the switching curve Γ, with $\sigma = (r_2, t_2)$

$$\left[\frac{b-c}{cr_2} - \lambda_1^1(t_2-)\right](-\alpha' r_2) + \left[-\frac{b}{c} - \lambda_2^1(t_2-)\right] = 0$$

(C

$$1 + \lambda_1^1(t_2-)r_2 + \lambda_2^1(t_2-)\,(u^1+cu^2) = 0$$

(C.12)

$$\left[\frac{b-c}{cr_2} - \lambda_1^2(t_2-)\right](-\alpha' r_2) + \left[-\frac{b}{c} - \lambda_2^1(t_2-)\right] = 0$$

$$|u^1| + b|u^2| + \lambda_1^2(t_2-)r_2 + \lambda_2^2(t_2-)\,(u^1+cu^2) = 0$$

Equations (C.12) can be solved consistent with (3.71) and (3.66) only under the imposed condition

$$2c - b^2 + 2bc + c^2 > 0 \qquad \text{(C.13)}$$

which can also be written as

$$\alpha' < \frac{1}{1+c} \qquad \text{(C.14)}$$

Then the result is that λ^2 is continuous and that

$$u^1(t) = u^2(t) = +1 \quad \text{for} \quad t < t_2 \qquad \text{(C.15)}$$

When the condition (C.13) is not met, there are no equilibrium sequences reaching Γ'.

Case (2) : c = b

In this case, the sequences $\begin{bmatrix} \pm 1 & \pm 1 & \mp 1 \\ \pm 1 & 0 & \mp \epsilon \end{bmatrix}$ with $0 \leq \epsilon \leq 1$ are in equilibrium. We prove this for the sequence given by the upper values. Figure C.1(ii) shows typical plots of the adjoint and control variables and

Also from (3.71) and (C.19), we have

$$t_3 - t_2 = \left(1 + \frac{c\epsilon}{2+c\epsilon}\right) \Big/ \left(\frac{2(1+c\epsilon)}{(2+c\epsilon)s_2}\right) = s_2 \qquad (C.21)$$

Solving the system equations (3.62) we get

$$s_2 = r_2 + (t_3 - t_2) = r_2 + s_2 \qquad (C.22)$$

Hence the switching curve Γ is given by

$$\Gamma = \left\{(r_1, r_2) : r_2 = 0\right\} \qquad (C.23)$$

A second application of the corner conditions at this corner proves the result.

Equation (C.18) also yields, under the imposed condition $\epsilon < \frac{c-2}{c}$, the following control law

$$u^1(t) = 1 \quad ; \quad u^2(t) = -1 \qquad \text{for} \quad t < t_3 \qquad (C.24)$$

The corresponding plot of the adjoint and control variables is not shown. It may be noted that the control law (C.24) is similar to (4.23) and can also be obtained similarly.

After obtaining these and other sequences exhaustively, the noncooperative solution is defined by a suitable selection of the sequences.

APPENDIX D

PROOF OF THE COOPERATIVE SOLUTION

For the Bicriterion Differential Game discussed in Chapters III, IV and V, we indicate here the proof of the Cooperative Solution. The game satisfies the equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u^1 + cu^2 \end{aligned} \tag{D.1}$$

with the Payoff Functionals defined as

$$\begin{aligned} J^1[s,u] &= \int_0^{t_f} dt \\ J^2[s,u] &= \int_0^{t_f} \{ |u^1| + b|u^2| \} \, dt \end{aligned} \tag{D.2}$$

The state at the final time t_f is specified as the origin.

In Section 5.5, for the cooperative solution, we considered the scalarized optimal control problems with the criterion functionals parametrized by k' as shown in (5.54). The application of the minimum principle to this problem yielded (5.55) - (5.57).

By considering a small ball (3.72) around the origin as we did in Example 3.2 and applying the transvarsality conditions (5.16), we get

$$[\lambda_1(t_f) \quad \lambda_2(t_f)] \begin{bmatrix} \delta \sin\theta & -\delta \sin\theta \\ u^1 + cu^2 & \delta \cos\theta \end{bmatrix} = [-\{k' + (1-k')(|u^1| + b|u^2|\} \quad 0] \tag{D.3}$$

Solving (D.3) and letting $\delta \to 0$, we have $\lambda_1(t_f)$ as arbitrary and

$$\lambda_2(t_f) = \frac{-k' + (1-k')(|u^1| + b|u^2|)}{(u^1 + cu^2)} \tag{D.4}$$

Now we obtain the optimal terminal sequences.

For $\begin{bmatrix} \pm 1 \\ \pm 1 \end{bmatrix}$ to be the terminal sequence, we should have from (5.57)

$$\frac{k' + (1-k')(1+b)}{1+c} > \frac{(1-k')b}{c}$$

$$\frac{k' + (1-k')(1+b)}{1+c} > (1-k') \tag{D.5}$$

Simplifying (D.5), we have either of the following :

(i) $c > b$ and $k' > \dfrac{c-b}{1-b+c}$

(ii) $c < b$ and $k' > \dfrac{b-c}{b}$ (D.6)

(iii) $c = b$

Similarly the following terminal sequences are optimal for the cases noted against them.

$$\begin{bmatrix} \pm 1 \\ 0 \end{bmatrix} \quad c < b \text{ and } k' < \frac{b-c}{b}$$

$$\begin{bmatrix} 0 \\ \pm 1 \end{bmatrix} \quad c > b \text{ and } k' < \frac{c-b}{1-b+c}$$

$$\begin{bmatrix} \pm 1 \\ \pm \epsilon \end{bmatrix} \quad c < b \text{ and } k' = \frac{b-c}{b} \tag{D.7}$$

$$\begin{bmatrix} \pm \epsilon \\ \pm 1 \end{bmatrix} \quad c > b \text{ and } k' = \frac{c-b}{1-b+c}$$

$$\begin{bmatrix} \pm \epsilon_1 \\ \pm \epsilon_2 \end{bmatrix} \quad c = b \text{ and } k' = 0$$

Now the various optimal sequences and the switching surfaces can be constructed as shown in Example 3.2. The procedure is even simpler in the present case since the adjoint variables are continuous. A typical plot of the adjoint variable λ_2 for the case $c > b$ and $k' > \frac{c-b}{1-b+c}$ is shown in Figure D.1. Let us denote the state at tt_3 and t_4 as (r_1, r_2) and (s_1, s_2) respectively. The corresponding switching surfaces, whose construction is indicated below, are shown in Figure 5.1(i).

The equation of the switching curve along which the state reaches the origin corresponding to Figure D.1 is obviously γ_{11}^- as defined in (3.83). Also, from Figure D.1, we have

$$t_4 - t_3 = \frac{(1-k') - \frac{b}{c}(1-k')}{\frac{k'+(1-k')(1+b)}{1+c}} (t_f - t_4) \qquad \text{(D.8)}$$

But from (A.6), (B.7) and (B.8), we have

$$t_f - t_4 = \frac{s_2}{1+c}$$

$$t_4 - t_3 = -\frac{r_2}{-c} + \frac{1}{-c}\sqrt{\frac{r_2^2 + 2r_1 c}{1-\frac{c}{1+c}}} \qquad \text{(D.9)}$$

$$s_2 = +\sqrt{\frac{r_2^2 + 2r_1 c}{1-\frac{c}{1+c}}}$$

If the switching curve Γ_o, on which (r_1, r_2) lies, is given by

$$\Gamma_o = \left\{(r_1, r_2) : r_1 = -\frac{\alpha_o}{2} r_2^2 \operatorname{sgn} r_2\right\} \qquad \text{(D.10)}$$

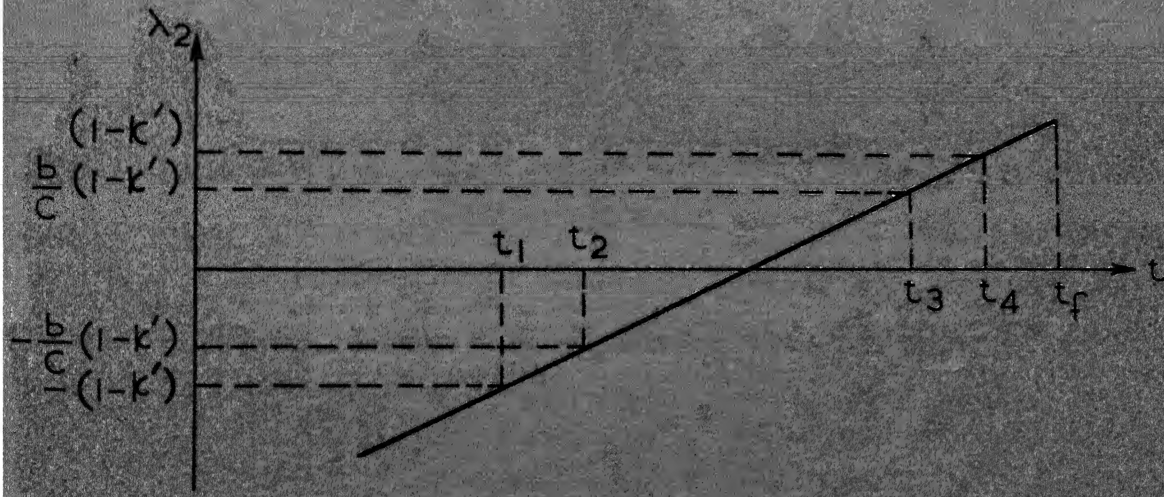


FIG. D.1 RESPONSE OF λ_2 CORRESPONDING TO OPTIMAL STRATEGY FOR THE COOPERATIVE SOLUTION CASE $c > b$ AND $k' > \frac{c-b}{1-b+c}$

The value of α_R is obtained by eliminating (r_1, r_2), (s_1, s_2), t_3, t_4 and t_f in (D.8) - (D.10). Thus we have

$$\alpha_R = \frac{k'^2 c + 2k'(1-k')(c-b) - (1-k')^2 (b-c)^2}{k'^2 c(1+c)} \tag{D.11}$$

By a similar technique the other switching curves given by

$$\begin{aligned} \Gamma_1 &= \left\{ (x_1, x_2) : x_1 = -\alpha_1 \frac{x_2^2}{2} \operatorname{sgn} x_2 \right\} \\ \Gamma_2 &= \left\{ (x_1, x_2) : x_1 = -\alpha_2 \frac{x_2^2}{2} \operatorname{sgn} x_2 \right\} \end{aligned} \tag{D.12}$$

are constructed. The complete cooperative solution of the problem is obtained by repeating this procedure for the other cases.

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